The Differential of a Map- HW Problems

1. Let $S^2_+ \subseteq \mathbb{R}^3$ be the upper hemisphere parametrized by

   $\vec{\Phi}(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right), \quad u^2+v^2<1,$

   and $S^2_- \subseteq \mathbb{R}^3$ the lower hemisphere parametrized by

   $\vec{\Psi}(\bar{u},\bar{v}) = \left(\bar{u}, \bar{v}, -\sqrt{1-\bar{u}^2-\bar{v}^2}\right), \quad \bar{u}^2+\bar{v}^2<1.$

   Let $f\colon S^2_+ \to S^2_-$ by $f(u, v, \sqrt{1-u^2-v^2})$=(-v,-u, -$\sqrt{1-u^2-v^2}$).

   a. Find the differential of f, df.

   b. At $u=\frac{1}{2}$, $v=\frac{\sqrt{2}}{2}$ let $\vec{w} \in T_{\vec{\Phi}\left(\frac{1}{2},\frac{\sqrt{2}}{2}\right)}S^2_+ = T_{\left(\frac{1}{2},\frac{\sqrt{2}}{2},\frac{1}{2}\right)}S^2_+$ be given as $\vec{w} = 2\vec{\Phi}_u\left(\frac{1}{2},\frac{\sqrt{2}}{2}\right) - \vec{\Phi}_v\left(\frac{1}{2},\frac{\sqrt{2}}{2}\right)$. Find $df(\vec{w})$ in terms of the basis $\frac{\partial\vec{\Psi}}{\partial\bar{u}}$ and $\frac{\partial\vec{\Psi}}{\partial\bar{v}}$ at the point $f\left(\vec{\Phi}\left(\frac{1}{2},\frac{\sqrt{2}}{2}\right)\right) = (-\frac{\sqrt{2}}{2}, -\frac{1}{2}, -\frac{1}{2})$ as well as finding it in terms of the standard basis in $\mathbb{R}^3$.

2. Let M be the surface in $\mathbb{R}^3$ given by

$$x^{-1}(x^1, x^2) = \left(x^1, x^2, (x^1)(x^2)\right); \qquad x^1, x^2 \in \mathbb{R}.$$

Let $N = S_+^2 = \{(x^1, x^2, x^3) \mid \ (x^1)^2 + (x^2)^2 + (x^3)^2 = 1, \ x^3 > 0\}$ be the upper hemisphere.

Define $\phi\colon M \to N$ by

$$\phi\left(x^1, x^2, (x^1)(x^2)\right) = \left(-\frac{x^2}{\sqrt{1+(x^1)^2+(x^2)^2}}, -\frac{x^1}{\sqrt{1+(x^1)^2+(x^2)^2}}, \frac{1}{\sqrt{1+(x^1)^2+(x^2)^2}}\right).$$

Let $y\colon N \to \mathbb{R}^2$ by $y\left(y^1, y^2, \sqrt{1-(y^1)^2-(y^2)^2}\right) = (y^1, y^2)$.

a. Find the differential of ϕ, $d\phi$.

b. Let $\vec{w} \in T_{(2,-2,-4)}M$ where $\vec{w} = \frac{\partial x^{-1}}{\partial x^1} - 2\,\frac{\partial x^{-1}}{\partial x^2}$. Find $d\phi(\vec{w})$ in the standard basis for $\mathbb{R}^3$.