Tangent Spaces

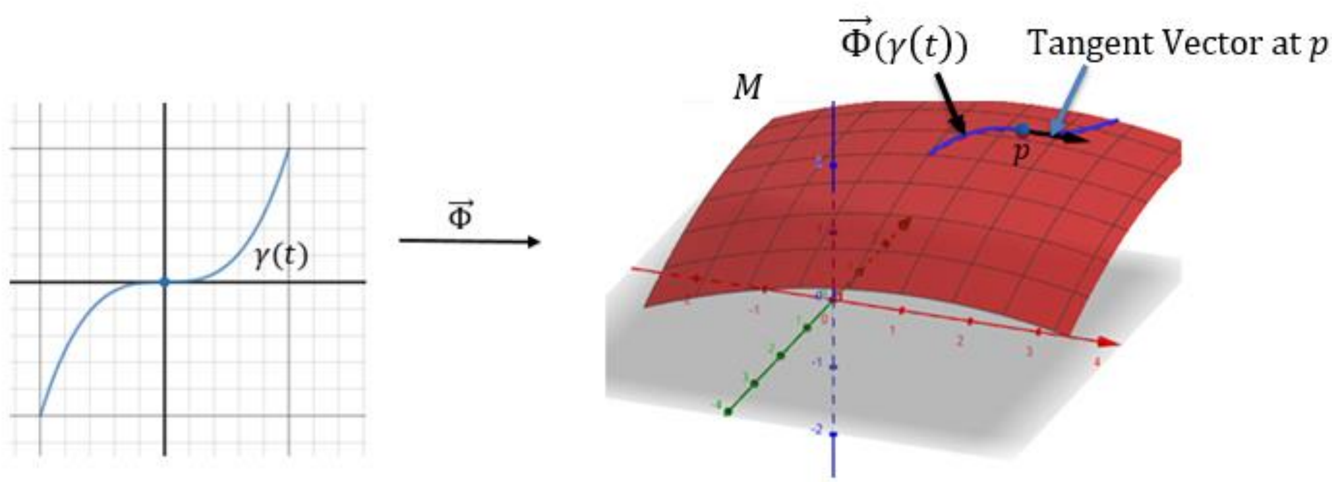
Let $M \subseteq \mathbb{R}^n$ be a k-dimensional manifold and $\overrightarrow{\Phi}$ a parameterization where $\overrightarrow{\Phi}: U \subseteq \mathbb{R}^k \to M \subseteq \mathbb{R}^n$ and $\overrightarrow{\Phi}(a) = x \in M$, then:

$$D\overrightarrow{\Phi}(a): \mathbb{R}^k_a \to \mathbb{R}^n_x \, .$$

Def. We call $D\overrightarrow{\Phi}(a)(\mathbb{R}^k_a) = T_x(M)$ the **tangent space** of M at x.

Note: this definition does not depend on the parameterization $\overrightarrow{\Phi}$.

Def. A **tangent vector** to a manifold, M, at a point $p \in M$, is the tangent vector at p of a curve in M passing through p.



Ex. Find a description of the tangent plane to the torus in $\mathbb{R}^4$ given by:

$$\vec{\Phi}(u,v) = (\cos u\,, \sin u\,, \cos v\,, \sin v)$$

at the point where: $(u,v) = \left(\frac{\pi}{6}, \frac{\pi}{4}\right)$.

$$D\vec{\Phi}(u,v) = \begin{pmatrix} -\sin u & 0 \\ \cos u & 0 \\ 0 & -\sin v \\ 0 & \cos v \end{pmatrix}$$

$$D\vec{\Phi}\left(\frac{\pi}{6}, \frac{\pi}{4}\right) = \begin{pmatrix} -\frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & 0 \\ 0 & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

The tangent space is spanned by the image of $< 1,0 >$ and $< 0,1 >$ under $D\vec{\Phi}\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$.

$$\left(D\vec{\Phi}\left(\frac{\pi}{6}, \frac{\pi}{4}\right)\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & 0 \\ 0 & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \\ 0 \end{pmatrix}$$

$$\left(D\vec{\Phi}\left(\frac{\pi}{6},\frac{\pi}{4}\right)\right)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & 0 \\ 0 & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}.$$

We want the tangent space at $\vec{\Phi}\left(\frac{\pi}{6},\frac{\pi}{4}\right) = \left(\frac{\sqrt{3}}{2},\frac{1}{2},\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$.

So we want all points of the form:

$$\vec{P}(s,t) = \left(\frac{\sqrt{3}}{2},\frac{1}{2},\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right) + \left(-\frac{1}{2},\frac{\sqrt{3}}{2},0,0\right)s + \left(0,0,\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)t$$

where $s,t \in \mathbb{R}$.

Suppose $\vec{\Phi}: U \subseteq \mathbb{R}^k \to M \subseteq \mathbb{R}^n$ is a parametrization of a k-dimensional manifold M. We know the standard basis for $T_p(\mathbb{R}^k)$, $p \in U$, is given by $(\vec{e}_1)_p, \ldots, (\vec{e}_k)_p$. What is the corresponding "standard" basis for $T_{\vec{\Phi}(p)}(M)$?

Ex. Suppose $\overrightarrow{\Phi}: U \subseteq \mathbb{R}^2 \to M \subseteq \mathbb{R}^3$ is a parametrization of a regular surface $M \subseteq \mathbb{R}^3$ given by:

$$\overrightarrow{\Phi}(x^1, x^2) = (x(x^1, x^2), y(x^1, x^2), z(x^1, x^2)).$$

By definition, the tangent space, $T_{\overrightarrow{\Phi}(p)}(M)$, is the image of:

$$D\overrightarrow{\Phi}(p): T_p(\mathbb{R}^2) \to T_{\overrightarrow{\Phi}(p)}(\mathbb{R}^3).$$

So to find the standard basis vectors for $T_{\overrightarrow{\Phi}(p)}(M)$, we need to find

$D\overrightarrow{\Phi}(p)(\vec{e}_1)_p$ and $D\overrightarrow{\Phi}(p)(\vec{e}_2)_p$.

To calculate this we just need to find the Jacobian matrix for $\overrightarrow{\Phi}(p)$:

$$D\overrightarrow{\Phi}(p) = \begin{pmatrix} \frac{\partial x}{\partial x^1} & \frac{\partial x}{\partial x^2} \\ \frac{\partial y}{\partial x^1} & \frac{\partial y}{\partial x^2} \\ \frac{\partial z}{\partial x^1} & \frac{\partial z}{\partial x^2} \end{pmatrix}.$$

Now we have:

$$D\overrightarrow{\Phi}(p)(\vec{e}_1)_p = \begin{pmatrix} \frac{\partial x}{\partial x^1} & \frac{\partial x}{\partial x^2} \\ \frac{\partial y}{\partial x^1} & \frac{\partial y}{\partial x^2} \\ \frac{\partial z}{\partial x^1} & \frac{\partial z}{\partial x^2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial x^1} \\ \frac{\partial y}{\partial x^1} \\ \frac{\partial z}{\partial x^1} \end{pmatrix} = \frac{\partial \overrightarrow{\Phi}}{\partial x^1} = \overrightarrow{\Phi}_{x^1}$$

$$D\vec{\Phi}(p)(\vec{e}_2)_p = \begin{pmatrix} \frac{\partial x}{\partial x^1} & \frac{\partial x}{\partial x^2} \\ \frac{\partial y}{\partial x^1} & \frac{\partial y}{\partial x^2} \\ \frac{\partial z}{\partial x^1} & \frac{\partial z}{\partial x^2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial x^2} \\ \frac{\partial y}{\partial x^2} \\ \frac{\partial z}{\partial x^2} \end{pmatrix} = \frac{\partial \vec{\Phi}}{\partial x^2} = \vec{\Phi}_{x^2}.$$

So the standard basis vectors for $T_{\vec{\Phi}(p)}(M)$ are given by $\frac{\partial \vec{\Phi}}{\partial x^1}$ and $\frac{\partial \vec{\Phi}}{\partial x^2}$ evaluated at p.

In general, for a k-dimensional manifold M in $\mathbb{R}^n$, parametrized by $\vec{\Phi}: U \subseteq \mathbb{R}^k \to M \subseteq \mathbb{R}^n$, the standard basis for $T_{\vec{\Phi}(p)}(M)$ is $\frac{\partial \vec{\Phi}}{\partial x^1}, \ldots, \frac{\partial \vec{\Phi}}{\partial x^k}$ evaluated at p.

Def. Let $U \subseteq \mathbb{R}^k$ be an open set with $p \in U$. Let $\vec{v} \in T_pU$ and $f: U \to \mathbb{R}$ any differentiable function. We can define a map from the set of real valued, continuously differentiable functions on $U, C^1(U, \mathbb{R})$, into the real numbers by taking a “kind” of **directional derivative** of f at $p \in U$ in the direction of $\vec{v}_p$ by:

$$\vec{v}_p(f) = < \frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^k} > \cdot < v_1, \ldots, v_k >$$

where $\frac{\partial f}{\partial x^i}$ is evaluated at p and $\vec{v}_p = v_1(\vec{e}_1)_p + \cdots + v_k(\vec{e}_k)_p$. This differs from the standard directional derivative because $\vec{v}_p$ is not necessarily a unit vector.

In particular if $\vec{v}_p = (\vec{e}_i)_p$, then:

$$(\vec{e}_i)_p(f) = < \frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^k} > \cdot < 0, \dots, 1, \dots, 0 > = \left.\frac{\partial f}{\partial x^i}\right|_p .$$

Thus we express $(\vec{e}_1)_p, \dots, (\vec{e}_k)_p$ as: $\left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^k}\right|_p$.

Ex. Let $f(x^1, x^2, x^3) = (x^1)^2 + (x^2)(x^3)$.
Let $p = (-2, 3, 4)$ and $\vec{v}_p = < -3, 2, 1 >$.
Find $\vec{v}_p(f)$.

$$\frac{\partial f}{\partial x^1} = 2x^1 \qquad \left.\frac{\partial f}{\partial x^1}\right|_p = -4$$

$$\frac{\partial f}{\partial x^2} = x^3 \qquad \left.\frac{\partial f}{\partial x^2}\right|_p = 4$$

$$\frac{\partial f}{\partial x^3} = x^2 \qquad \left.\frac{\partial f}{\partial x^3}\right|_p = 3$$

$$(\vec{v}_p)(f) = < -4, 4, 3 > \cdot < -3, 2, 1 > = 12 + 8 + 3 = 23.$$

We can do the same calculation for manifolds.

If $\overrightarrow{\Phi}: U \subseteq \mathbb{R}^k \to M \subseteq \mathbb{R}^n$, $p \in M$, $\vec{v}_p \in T_pM$, and $f: M \to \mathbb{R}$, then

$$\vec{v}_p(f) = < \frac{\partial f}{\partial u^1}, \dots, \frac{\partial f}{\partial u^k} > \cdot < v_1, \dots, v_k >$$

where $(u^1, \dots, u^k)$ are coordinates on M and:

$$\vec{v}_p = v_1\overrightarrow{\Phi}_{u^1} + \cdots + v_k\overrightarrow{\Phi}_{u^k} .$$

Ex. Let $\overrightarrow{\Phi}(u^1, u^2) = \big((u^1), (u^2), (u^1)^2 + (u^2)^2\big)$.
Let $(u^1, u^2) = (1, 2)$ so $p = \overrightarrow{\Phi}(1, 2) = (1, 2, 5)$.
Let $\vec{v}_p = 3\overrightarrow{\Phi}_{u^1}(1, 2) - 2\overrightarrow{\Phi}_{u^2}(1, 2)$.
Let $f(x, y, z) = x^4 + y^3 + z^2$. Find $\vec{v}_p(f)$.

We can do this calculation in u^1, u^2 or in x, y, z.

In u^1, u^2: $\quad (\vec{v}_p)(f) = < \frac{\partial f}{\partial u^1}, \frac{\partial f}{\partial u^2} > \cdot < 3, -2 >$

$$x = u^1 \qquad y = u^2 \qquad z = (u^1)^2 + (u^2)^2$$

$$\frac{\partial x}{\partial u^1} = 1 \qquad \frac{\partial y}{\partial u^1} = 0 \qquad \frac{\partial z}{\partial u^1} = 2u^1$$

$$\frac{\partial x}{\partial u^2} = 0 \qquad \frac{\partial y}{\partial u^2} = 1 \qquad \frac{\partial z}{\partial u^2} = 2u^2 .$$

At p: $x = 1$, $y = 2$, $z = 5$, $u^1 = 1$, $u^2 = 2$, so we can write:

$$\frac{\partial f}{\partial u^1} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u^1} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u^1} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u^1} = 4x^3(1) + 2z(2u^1)$$

$$\frac{\partial f}{\partial u^1} = 4 + 2(5)(2)(1) = 24$$

$$\frac{\partial f}{\partial u^2} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u^2} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u^2} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u^2} = 3y^2(1) + 2z(2u^2) = 52.$$

$$(\vec{v}_p)(f) = < 24, 52 > \cdot < 3, -2 > = 72 - 104 = -32.$$

In x, y, z:

$$\vec{v}_p = 3\overrightarrow{\Phi}_{u^1}(1,2) - 2\overrightarrow{\Phi}_{u^2}(1,2) = 3 < 1,0,2 > -2 < 0,1,4 >$$
$$= < 3,-2,-2 > .$$

$$\vec{v}_p(f) = < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} > \cdot < 3,-2,-2 >$$

$$\frac{\partial f}{\partial x} = 4x^3 \qquad \left.\frac{\partial f}{\partial x}\right|_p = 4$$

$$\frac{\partial f}{\partial y} = 3y^2 \qquad \left.\frac{\partial f}{\partial y}\right|_p = 12$$

$$\frac{\partial f}{\partial z} = 2z \qquad \left.\frac{\partial f}{\partial z}\right|_p = 10$$

$$\vec{v}_p(f) = < 4, 12, 10 > \cdot < 3,-2,-2 > = 12 - 24 - 20 = -32.$$

Def. A **vector field** on a manifold, M, assigns to each point $x \in M$ a vector in $T_x M$.

Thus, we could let a vector field, $\vec{v}_x$, on M, map a real valued function, f, on M into a real valued function, g, on M, by $g(x) = \vec{v}_x(f)$.