Manifolds

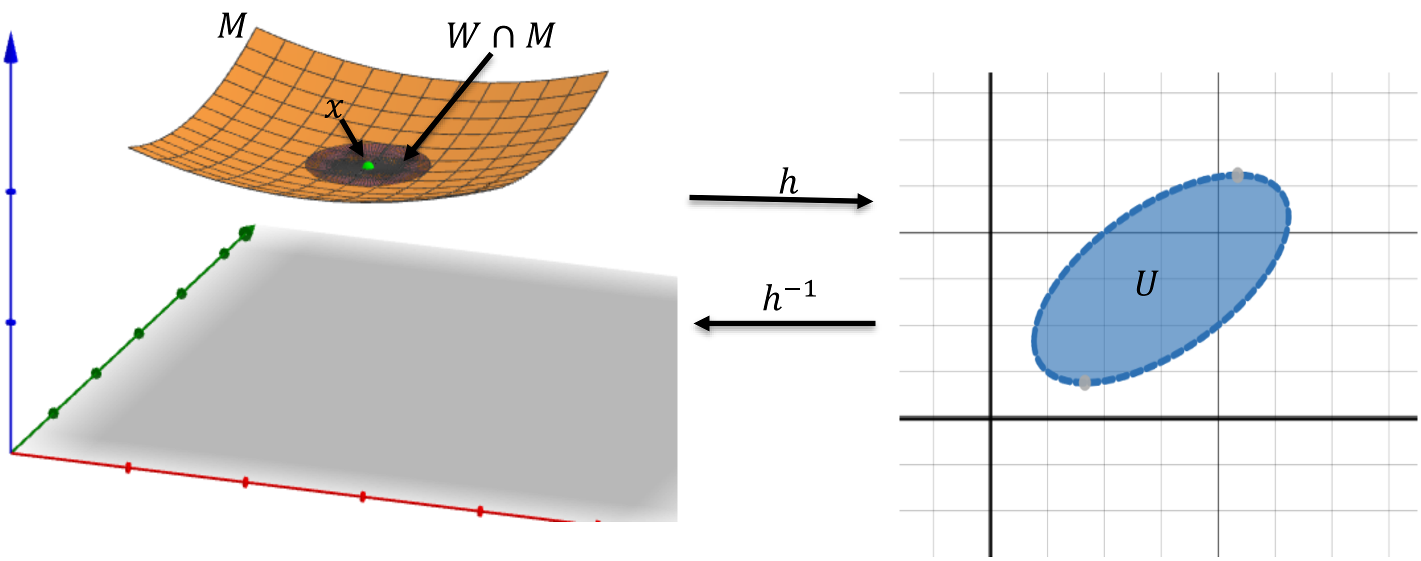
Def. Let U and V be open sets in $\mathbb{R}^n$. A differentiable function, $h: U \to V$ with a differentiable inverse $h^{-1}: V \to U$, is called a **diffeomorphism** ("differentiable" will mean C^∞ from here on).

Def. A subset, $M \subseteq \mathbb{R}^n$, is called a **differentiable manifold** (or just a manifold) of dimension k if for each point $x \in M$ there is an open set $W \subseteq \mathbb{R}^n$, an open set $U \subseteq \mathbb{R}^k$, and a diffeomorphism:

$$h: W \cap M \to U.$$

h is called a **system of coordinates** on $W \cap M$.

$h^{-1}: U \to W \cap M$ is called a **parameterization** of $W \cap M$.



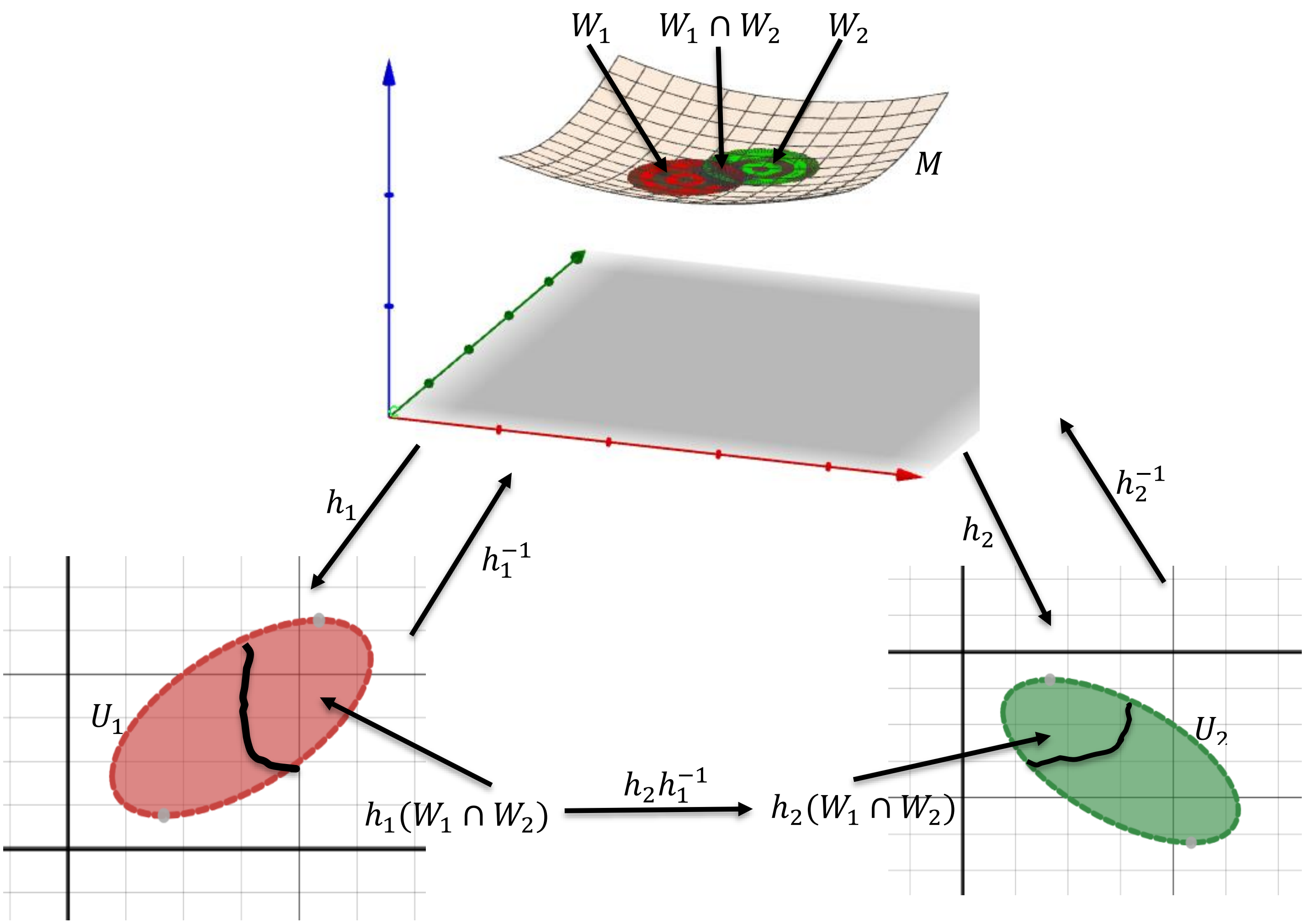
The set $\{h_\alpha, W_\alpha\}$ of coordinate functions and sets W_α that cover M is called an **atlas**.

Ex. A point in $\mathbb{R}^n$ is a zero dimensional manifold.
An open set in $\mathbb{R}^n$ is an n-dimensional manifold.

Notice that if (h_1, W_1) and (h_2, W_2) are two coordinate systems on $W_1, W_2 \subseteq M$, where $h_1: W_1 \to U_1$ and $h_2: W_2 \to U_2$, then:

$$h_{12} = h_2 h_1^{-1}: h_1(W_1 \cap W_2) \to h_2(W_1 \cap W_2)$$

is a differentiable map of an open set in $\mathbb{R}^k$ *into an open set in* $\mathbb{R}^k$, and is called a **transition function** between the coordinate systems (h_1, W_1) and (h_2, W_2).



Def. An atlas (h_α, W_α) is called **smooth** if all of the transition functions are smooth.

Ex. Show that $S^2 = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ is a (differentiable) manifold.

One way to do this is to define the following 6 parameterizations of the sphere, which cover the entire sphere.

$$\vec{\Phi}_i : V \to \mathbb{R}^3 \text{ where } V = \{(u,v) \in \mathbb{R}^2 |\, u^2 + v^2 < 1\}$$

$$\vec{\Phi}_1(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right) \qquad (z > 0)$$

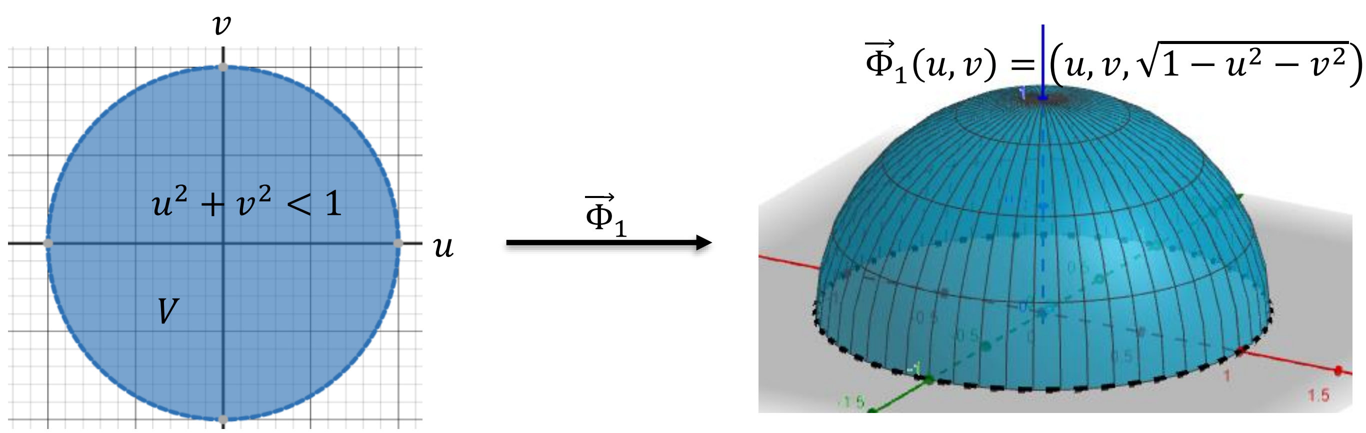
$$\vec{\Phi}_2(u,v) = \left(u, v, -\sqrt{1-u^2-v^2}\right) \qquad (z < 0)$$

$$\vec{\Phi}_3(u,v) = \left(u, \sqrt{1-u^2-v^2}, v\right) \qquad (y > 0)$$

$$\vec{\Phi}_4(u,v) = \left(u, -\sqrt{1-u^2-v^2}, v\right) \qquad (y < 0)$$

$$\vec{\Phi}_5(u,v) = \left(\sqrt{1-u^2-v^2}, u, v\right) \qquad (x > 0)$$

$$\vec{\Phi}_6(u,v) = \left(-\sqrt{1-u^2-v^2}, u, v\right) \qquad (x < 0)$$



To show that these 6 parameterizations make S^2 into a manifold we must show:

1) $\vec{\Phi}_i$ is a diffeomorphism, for $i = 1, \dots, 6$
2) $\bigcup_{i=1}^{6} \vec{\Phi}_i(V) \supseteq S^2$.

To show that $\overrightarrow{\Phi}_i$ is a diffeomorphism we must show:

a. $\overrightarrow{\Phi}_i$ is one to one
b. $\overrightarrow{\Phi}_i$ is onto its image
c. $\overrightarrow{\Phi}_i$ and $\overrightarrow{\Phi}_i^{-1}$ are differentiable.

Let's show that $\overrightarrow{\Phi}_1$ is a diffeomorphism.

a.
$$\overrightarrow{\Phi}_1(u,v) = \overrightarrow{\Phi}_1(u',v')$$
$$\left(u, v, \sqrt{1-u^2-v^2}\right) = \left(u', v', \sqrt{1-u'^2-v'^2}\right)$$
So $(u,v) = (u',v')$ and $\overrightarrow{\Phi}_1$ is one to on.

b. By definition $\overrightarrow{\Phi}_1$ maps V onto $\overrightarrow{\Phi}_1(V)$.

c. Each $\overrightarrow{\Phi}_i$ is differentiable on V because all of the partial derivatives of all orders exist (since $u^2+v^2 \neq 1$). The inverse functions of the $\overrightarrow{\Phi}_i$s are just projections. For example:

$$\left(\overrightarrow{\Phi}_1\right)^{-1}\left(u, v, \sqrt{1-u^2-v^2}\right) = (u,v).$$

All partial derivatives of all orders exist so $\left(\overrightarrow{\Phi}_1\right)^{-1}$ is differentiable. The same holds for the other $\left(\overrightarrow{\Phi}_i\right)^{-1}$.

$\bigcup_{i=1}^{6} \overrightarrow{\Phi}_i(V) \supseteq S^2$ because every point of S^2 has at least one non-zero coordinate.

What do the transition functions look like? First, notice that not all $\vec{\Phi}_i(V), \vec{\Phi}_j(V)$ intersect (e.g. $\vec{\Phi}_1(V)$ is the upper hemisphere and $\vec{\Phi}_2(V)$ is the lower hemisphere). As an example, let's look at $\vec{\Phi}_1(V) \cap \vec{\Phi}_3(V)$.

$\vec{\Phi}_1(V) =$ points on S^2 with $z > 0$
$\vec{\Phi}_3(V) =$ points on S^2 with $y > 0$

$\vec{\Phi}_1(V) \cap \vec{\Phi}_3(V) =$ points on S^2 with $y > 0$ and $z > 0$.

$\vec{\Phi}_3(u, v) = \left(u, \sqrt{1 - u^2 - v^2}, v\right)$
$\vec{\Phi}_3^{-1}\left(u, \sqrt{1 - u^2 - v^2}, v\right) = (u, v)$.

So $\left(\vec{\Phi}_3\right)^{-1} \vec{\Phi}_1(u, v) = \vec{\Phi}_3^{-1}\left(u, v, \sqrt{1 - u^2 - v^2}\right) = \left(u, \sqrt{1 - u^2 - v^2}\right)$.

Other transition functions are also differentiable, thus $\left\{\vec{\Phi}_i^{-1}, \vec{\Phi}_i(V)\right\}$ for $i = 1, \dots, 6$ is a smooth atlas for S^2.

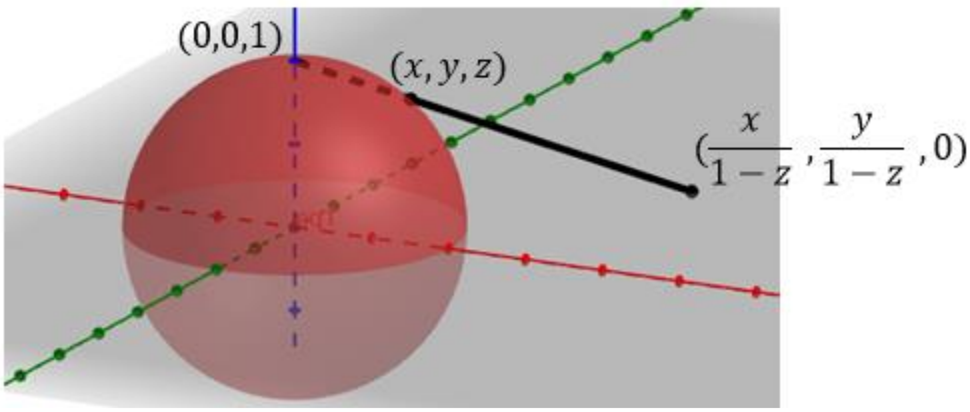
Stereographic Projection:

A second way to show that the unit sphere $S^2 \subseteq \mathbb{R}^3$ is a manifold is by using a stereographic projection. We start by covering S^2 with two open sets:

$$W_1 = S^2 - (0, 0, 1)$$

$$W_2 = S^2 - (0, 0, -1).$$

To map W_1 onto $\mathbb{R}^2$ we take any point $(x, y, z) \in S^2 - (0,0,1)$ and imagine a line, l, through the points (x, y, z) and $(0,0,1)$ (the north pole). We will map (x, y, z) to the intersection of the line, l, with the x, y-plane.



Let's find a formula for this point of intersection. A direction vector for this line is given by: $< x, y, z > - < 0,0,1 > = < x, y, z-1 >$. So a vector equation of the line is given by:

$$l(t) = < 0,0,1 > + t < x, y, z-1 > = < tx, ty, t(z-1)+1 >$$

where $t \in \mathbb{R}$.

This line intersects the x, y-plane when $t(z-1)+1 = 0$ or $t = \frac{1}{1-z}$.

So the point of intersection with the x, y-plane is: $< \frac{x}{1-z}, \frac{y}{1-z} >$.

Thus, $\pi_1: W_1 \to \mathbb{R}^2$ by $\pi_1(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$.

Next we want to show that π_1 is a diffeomorphism.

Claim: π_1 is one-to-one.

$$\pi_1(x_1, y_1, z_1) = \pi_1(x_2, y_2, z_2)$$

$$\Longrightarrow \quad \frac{x_1}{1-z_1} = \frac{x_2}{1-z_2} \;\; ; \;\; \frac{y_1}{1-z_1} = \frac{y_2}{1-z_2}\,.$$

But notice:

$$\left(\frac{x_1}{1-z_1}\right)^2 + \left(\frac{y_1}{1-z_1}\right)^2 + 1 = \frac{x_1^2+y_1^2+(1-z_1)^2}{(1-z_1)^2}$$

$$= \frac{x_1^2+y_1^2+z_1^2-2z_1+1}{(1-z_1)^2} = \frac{2(1-z_1)}{(1-z_1)^2} = \frac{2}{1-z_1}\,.$$

By the same argument, $\left(\frac{x_2}{1-z_2}\right)^2 + \left(\frac{y_2}{1-z_2}\right)^2 + 1 = \frac{2}{1-z_2}\,.$

But then $\frac{2}{1-z_1} = \frac{2}{1-z_2} \;\; \Rightarrow z_1 = z_2.$ Thus:

$$\frac{x_1}{1-z_1} = \frac{x_2}{1-z_2} \quad \Rightarrow x_1 = x_2$$

$$\frac{y_1}{1-z_1} = \frac{y_2}{1-z_2} \quad \Rightarrow y_1 = y_2.$$

And so π_1 is one-to-one.

Claim: π_1 is onto $\mathbb{R}^2$.

Let $(a, b) \in \mathbb{R}^2$, we must find x, y, z such that $\pi_1(x, y, z) = (a, b)$ and $(x, y, z) \in W_1$.

$$\frac{x}{1-z} = a$$

$$\frac{y}{1-z} = b.$$

As before:

$$\frac{2}{1-z} = \left(\frac{x}{1-z}\right)^2 + \left(\frac{y}{1-z}\right)^2 + 1 = a^2 + b^2 + 1$$

$$\Longrightarrow \qquad 1 - z = \frac{2}{a^2+b^2+1}.$$

But:

$$\frac{x}{1-z} = a \;\Rightarrow x = a(1-z) = \frac{2a}{a^2+b^2+1}$$

$$\frac{y}{1-z} = b \;\Rightarrow y = b(1-z) = \frac{2b}{a^2+b^2+1}$$

$$z = 1 - \frac{2}{a^2+b^2+1} = \frac{a^2+b^2-1}{a^2+b^2+1}.$$

$$\Longrightarrow \qquad \pi_1\left(\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1}\right) = (a, b).$$

How do we know $p = \left(\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1}\right) \in W_1$?

$$\left(\frac{2a}{a^2+b^2+1}\right)^2 + \left(\frac{2b}{a^2+b^2+1}\right)^2 + \left(\frac{a^2+b^2-1}{a^2+b^2+1}\right)^2 = \frac{\left(a^2+b^2+1\right)^2}{(a^2+b^2+1)^2} = 1$$

So $p \in S^2$.

$a^2 + b^2 - 1 < a^2 + b^2 + 1$ so we know $z = \frac{a^2+b^2-1}{a^2+b^2+1} \neq 1$

Thus, $p \in W_1$.

In fact, we just showed that $\pi_1^{-1}: \mathbb{R}^2 \to S^1 - (0,0,1)$ by:

$$\pi_1^{-1}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right).$$

Partial derivatives of all orders exist for π_1 and π_1^{-1}, so they are both C^∞. Thus, π_1 is a diffeomorphism.

Similarly, $\pi_2: S^2 - (0,0,-1) \to \mathbb{R}^2$ by: $\pi_2(x,y,z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$ is also a diffeomorphism.

Notice that $\pi_1^{-1}(\mathbb{R}^2) \cup \pi_2^{-1}(\mathbb{R}^2) \supseteq S^2$ since:

$$\pi_1^{-1}(\mathbb{R}^2) = S^2 - (0,0,1)$$

$$\pi_2^{-1}(\mathbb{R}^2) = S^2 - (0,0,-1).$$

Thus S^2 is a smooth manifold.

Calculating the transition function $\pi_2\pi_1^{-1}(u,v)$ we get:

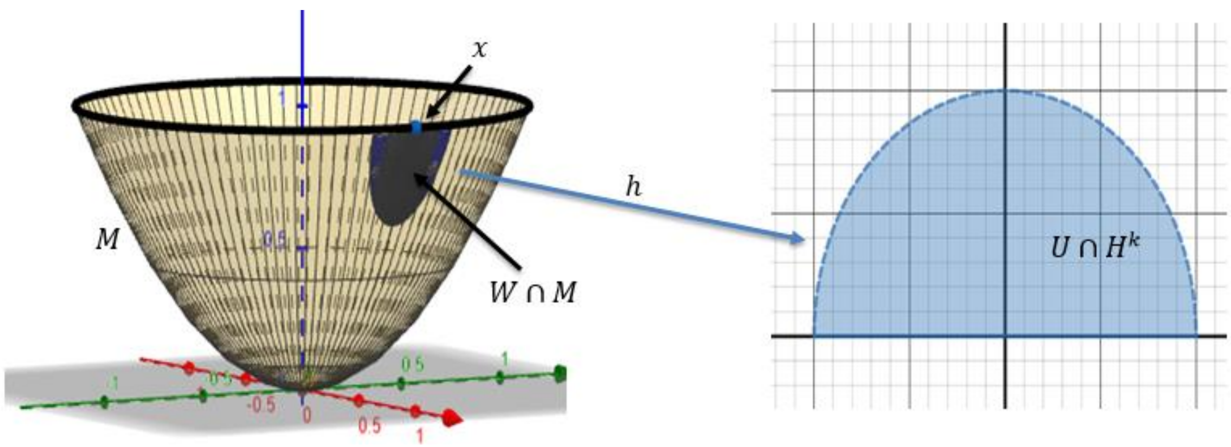
$$\pi_2\pi_1^{-1}(u,v) = \pi_2 \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right) = \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right).$$

Def. $H^k = \{x \in \mathbb{R}^k \big| x_k \geq 0\}$ is called the **half-space**.

Ex. H^2 is the upper half plane with the x-axis and

$$H^3 = \{(x, y, z) \in \mathbb{R}^3 |\ z \geq 0\}.$$

Def. $M \subseteq \mathbb{R}^n$ is a ***k*-dimensional manifold with boundary** if each $x \in M$ has a neighborhood $W \cap M$ that is diffeomorphic to an open set $U \subseteq \mathbb{R}^k$ or diffeomorphic to $U \cap H^k$, where U is an open set in $\mathbb{R}^k$. The set of points in M where $W \cap M$ is diffeomorphic to $U \cap H^k$ are called **boundary points** of M.

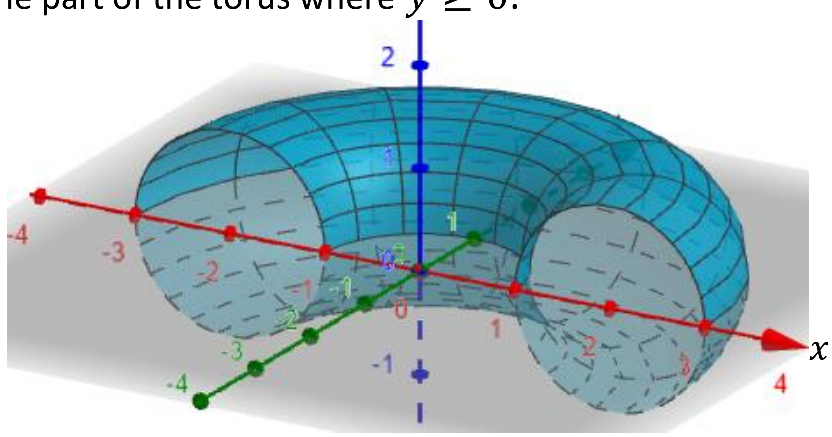


Ex. An example of a manifold with a boundary is the half torus, M, in $\mathbb{R}^3$ given by:

$$\vec{\Phi}(u, v) = \big((2 + \cos v) \cos u\, , (2 + \cos v) \sin u\, , \sin v\big)$$

where $(u, v) \in [0, \pi] \times [0, 2\pi]$.

This is the part of the torus where $y \geq 0$.



We can cover this manifold with 4 sets:

$$W_1 \cap M = \{p \in M | 0 < v < \pi, \ \ 0 \le u \le \pi\}$$

$$W_2 \cap M = \left\{p \in M \middle| \frac{\pi}{2} < v < \frac{3\pi}{2}, \ \ 0 \le u \le \pi\right\}$$

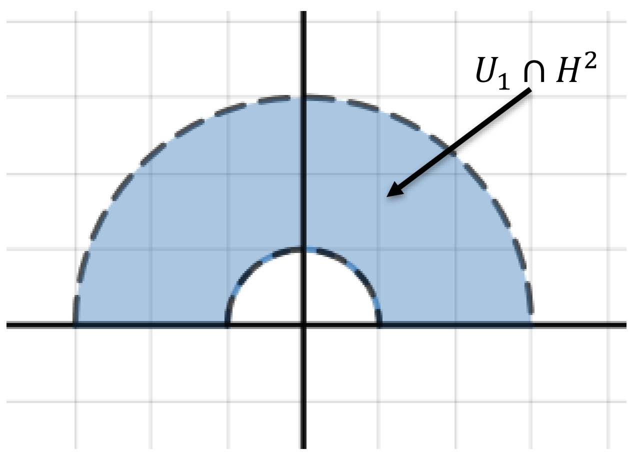
$$W_3 \cap M = \{p \in M | \pi < v < 2\pi, \ \ 0 \le u \le \pi\}$$

$$W_4 \cap M = \left\{p \in M \middle| \frac{3\pi}{2} < v \le 2\pi \text{ or } 0 < v \le \frac{\pi}{2}, \ \ 0 \le u \le \pi\right\}$$

Let $U_1 = \{(x, y) | \ 1 < x^2 + y^2 < 9\}$.

Define: $h_1 : W_1 \cap M \to U_1 \cap H^2$
by $h_1(x, y, z) = (x, y)$.

This is just the projection of M where $z > 0$ onto part of an annulus in the x, y-plane.



It's not hard to show that h_1 is a diffeomorphism. h_3 is the same function except it maps points in M where $z < 0$ on to $U_1 \cap H^2$.

h_2 and h_4 can be gotten by first rotating $W_2 \cap M$ and $W_4 \cap M$ by $\frac{\pi}{2}$ (i.e. replace v with $v - \frac{\pi}{2}$) and then apply h_1 and h_3 respectively.

Def. Let M be a differentiable manifold of dimension k. We say M is orientable if there is an atlas for $M, \{h_\alpha, W_\alpha\}$, such that all of the transition functions: $h_\beta \circ h_\alpha^{-1}: h_\alpha(W_\alpha \cap W_\beta) \to h_\beta(W_\alpha \cap W_\beta)$ have positive Jacobians (i.e. $\det\left(\left(h_\beta \circ h_\alpha^{-1}\right)'\right) > 0$).

Ex. Consider the following atlas on S^2

$$\pi_1: S^2 - (0,0,1) \to \mathbb{R}^2$$

$$\pi_1(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

$$\pi_2: S^2 - (0,0,-1) \to \mathbb{R}^2$$

$$\pi_2(x,y,z) = \left(\frac{x}{1+z}, -\frac{y}{1+z}\right).$$

From a homework problem you will see that:

$$\pi_1^{-1}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right).$$

Thus we have:

$$(\pi_2 \circ \pi_1^{-1})(u,v) = \pi_2\left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$$

$$= \left(\frac{u}{u^2+v^2}, \frac{-v}{u^2+v^2}\right).$$

$$(\pi_2 \circ \pi_1^{-1})'(u,v) = \begin{pmatrix} \frac{u^2-v^2}{(u^2+v^2)^2} & \frac{-2uv}{(u^2+v^2)^2} \\ \frac{2uv}{(u^2+v^2)^2} & \frac{u^2-v^2}{(u^2+v^2)^2} \end{pmatrix}$$

$$\det((\pi_2 \circ \pi_1^{-1})') = \frac{\left(u^2-v^2\right)^2+4u^2v^2}{(u^2+v^2)^4} = \frac{1}{(u^2+v^2)^2} > 0$$

and finite since $\pi_1^{-1}(0,0) = (0,0,-1)$, which is not part of the domain of π_2. Thus we can say S^2 is orientable.

Note: the atlas with $\pi_1(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$ and $\pi_2(x,y,z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$; the standard stereographic projection does not have:

$$\det((\pi_2 \circ \pi_1^{-1})') > 0.$$