Manifolds- HW Problems

1. Let $S^3 = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 |\, \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 + \left(x^4\right)^2 = 1\}$.

   Let $W_1 = S^3 - (0,0,0,1)$; and $\pi_1 : W_1 \to \mathbb{R}^3$ by

   $$\pi_1(x^1, x^2, x^3, x^4) = \left(\frac{x^1}{1-x^4}, \frac{x^2}{1-x^4}, \frac{x^3}{1-x^4}\right).$$

   Let $W_2 = S^3 - (0,0,0,-1)$; and $\pi_2 : W_2 \to \mathbb{R}^3$ by

   $$\pi_2(x^1, x^2, x^3, x^4) = \left(\frac{x^1}{1+x^4}, \frac{x^2}{1+x^4}, \frac{x^3}{1+x^4}\right).$$

   Show that (π_1, W_1), (π_2, W_2) is a smooth atlas for S^3 by showing

   a. π_1 and π_2 are diffeomorphisms.

   b. $W_1 \cup W_2 \supseteq S^3$

   c. $\pi_2 \circ \pi_1^{-1}$ is C^∞. Find $\pi_2 \circ \pi_1^{-1}$ first.

2. Let $S^1 = \{(x, y) \in \mathbb{R}^2 | \, x^2 + y^2 = 1\}$.

   Let $W_1 = S^1 - (-1, 0)$, $\quad W_2 = S^1 - (1,0)$ and

   $y_1 \colon \mathbb{R} \to S^1 - (-1,0)$ by $y_1(t) = (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$

   $y_2 \colon \mathbb{R} \to S^1 - (1,0)$ by $y_2(t) = (\frac{t^2-1}{1+t^2}, \frac{-2t}{1+t^2})$.

   Show that:

   a. y_1 and y_2 are diffeomorphisms (you might want to use the fact $\frac{1-t^2}{1+t^2} = -1 + \frac{2}{1+t^2}$).

   b. $W_1 \cup W_2 \supseteq S^1$.

   c. $y_1^{-1} y_2$ is C^∞.