Higher Order Tensors

Def. A set of components $T_{ij}, i, j = 1, \dots, n$ is said to be a **tensor of type** $(\mathbf{0}, \mathbf{2})$ at a point $p \in \mathbb{R}^n$ if under a change of coordinates the components transform according to:

$$\bar{T}_{kl} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^l} T_{ij} \, .$$

Suppose a surface, $S \subseteq \mathbb{R}^3$, is given by:

$$\vec{\Phi}(x^1, x^2) = \big(f_1(x^1, x^2),\ \ f_2(x^1, x^2\ ),\ \ \ f_3(x^1, x^2)\big).$$

A surface in $\mathbb{R}^3$ is called regular if $\vec{\Phi}_{x^1} \times \vec{\Phi}_{x^2} \neq 0$ (This guarantees existence of a tangent plane.) where $\vec{\Phi}_{x^i} = \frac{\partial \vec{\Phi}}{\partial x^i}$.

Define: $$g_{ij} = \vec{\Phi}_{x^i} \cdot \vec{\Phi}_{x^j}$$

(g_{ij}) is called the metric tensor of $\vec{\Phi}$.

Ex. Show that (g_{ij}) is a $(0, 2)$ tensor.

$$\bar{g}_{ij} = \vec{\Phi}_{\bar{x}^i} \cdot \vec{\Phi}_{\bar{x}^j} = \frac{\partial \vec{\Phi}}{\partial \bar{x}^i} \cdot \frac{\partial \vec{\Phi}}{\partial \bar{x}^j}$$

$$\frac{\partial \vec{\Phi}}{\partial \bar{x}^i} = \frac{\partial \vec{\Phi}}{\partial x^1} \frac{\partial x^1}{\partial \bar{x}^i} + \frac{\partial \vec{\Phi}}{\partial x^2} \frac{\partial x^2}{\partial \bar{x}^i}$$

$$\frac{\partial \vec{\Phi}}{\partial \bar{x}^j} = \frac{\partial \vec{\Phi}}{\partial x^1} \frac{\partial x^1}{\partial \bar{x}^j} + \frac{\partial \vec{\Phi}}{\partial x^2} \frac{\partial x^2}{\partial \bar{x}^j}$$

$$\bar{g}_{ij} = \vec{\Phi}_{\bar{x}^i} \cdot \vec{\Phi}_{\bar{x}^j} = \left(\frac{\partial \vec{\Phi}}{\partial x^1}\frac{\partial x^1}{\partial \bar{x}^i} + \frac{\partial \vec{\Phi}}{\partial x^2}\frac{\partial x^2}{\partial \bar{x}^i}\right) \cdot \left(\frac{\partial \vec{\Phi}}{\partial x^1}\frac{\partial x^1}{\partial \bar{x}^j} + \frac{\partial \vec{\Phi}}{\partial x^2}\frac{\partial x^2}{\partial \bar{x}^j}\right)$$

$$= \left(\frac{\partial \vec{\Phi}}{\partial x^1} \cdot \frac{\partial \vec{\Phi}}{\partial x^1}\right)\frac{\partial x^1}{\partial \bar{x}^i}\frac{\partial x^1}{\partial \bar{x}^j} + \left(\frac{\partial \vec{\Phi}}{\partial x^1} \cdot \frac{\partial \vec{\Phi}}{\partial x^2}\right)\frac{\partial x^1}{\partial \bar{x}^i}\frac{\partial x^2}{\partial \bar{x}^j}$$

$$+ \left(\frac{\partial \vec{\Phi}}{\partial x^2} \cdot \frac{\partial \vec{\Phi}}{\partial x^1}\right)\frac{\partial x^2}{\partial \bar{x}^i}\frac{\partial x^1}{\partial \bar{x}^j} + \left(\frac{\partial \vec{\Phi}}{\partial x^2} \cdot \frac{\partial \vec{\Phi}}{\partial x^2}\right)\frac{\partial x^2}{\partial \bar{x}^i}\frac{\partial x^2}{\partial \bar{x}^j}$$

$$= g_{11}\frac{\partial x^1}{\partial \bar{x}^i}\frac{\partial x^1}{\partial \bar{x}^j} + g_{12}\frac{\partial x^1}{\partial \bar{x}^i}\frac{\partial x^2}{\partial \bar{x}^j} + g_{21}\frac{\partial x^2}{\partial \bar{x}^i}\frac{\partial x^1}{\partial \bar{x}^j} + g_{22}\frac{\partial x^2}{\partial \bar{x}^i}\frac{\partial x^2}{\partial \bar{x}^j}$$

$$\bar{g}_{ij} = \sum_{k=1}^{2}\sum_{l=1}^{2} g_{kl}\frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j}.$$

So (g_{ij}) is a $(0, 2)$ tensor.

Ex. Let $\vec{\Phi}(x^1, x^2) = (x^1, x^2, (x^1)^2 + (x^2)^2)$.

a) Find g_{ij} with respect to (x^1, x^2)

b) Let $x^1 = \bar{x}^1 \cos \bar{x}^2$ and $x^2 = \bar{x}^1 \sin \bar{x}^2$. Find $\bar{g}_{kl}$ by using:

$$\bar{g}_{kl} = \sum_{j=1}^{2}\sum_{i=1}^{2} g_{ij}\frac{\partial x^i}{\partial \bar{x}^k}\frac{\partial x^j}{\partial \bar{x}^l}.$$

a) $\vec{\Phi}_{x^1} = \frac{\partial \vec{\Phi}}{\partial x^1} = (1, 0, 2x^1)$ and $\vec{\Phi}_{x^2} = \frac{\partial \vec{\Phi}}{\partial x^2} = (0, 1, 2x^2)$

$$g_{11} = \vec{\Phi}_{x^1} \cdot \vec{\Phi}_{x^1} = 1 + 4(x^1)^2$$

$$g_{12} = g_{21} = \vec{\Phi}_{x^1} \cdot \vec{\Phi}_{x^2} = 4x^1x^2$$

$$g_{22} = \vec{\Phi}_{x^2} \cdot \vec{\Phi}_{x^2} = 1 + 4(x^2)^2$$

$$(g_{ij}) = \begin{pmatrix} 1 + 4(x^1)^2 & 4x^1x^2 \\ 4x^1x^2 & 1 + 4(x^2)^2 \end{pmatrix}.$$

b) Find $\bar{g}_{kl}$

$$\bar{g}_{kl} = \sum_{j=1}^{2}\sum_{i=1}^{2} g_{ij} \frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^l}.$$

For $\bar{g}_{11}$, $k = 1$, $l = 1$ so:

$$\bar{g}_{11} = \sum_{j=1}^{2}\sum_{i=1}^{2} g_{ij} \frac{\partial x^i}{\partial \bar{x}^1} \frac{\partial x^j}{\partial \bar{x}^1}$$

$$= g_{11} \frac{\partial x^1}{\partial \bar{x}^1} \frac{\partial x^1}{\partial \bar{x}^1} + g_{12} \frac{\partial x^1}{\partial \bar{x}^1} \frac{\partial x^2}{\partial \bar{x}^1} + g_{21} \frac{\partial x^2}{\partial \bar{x}^1} \frac{\partial x^2}{\partial \bar{x}^1} + g_{22} \frac{\partial x^2}{\partial \bar{x}^1} \frac{\partial x^2}{\partial \bar{x}^1}$$

$$\frac{\partial x^1}{\partial \bar{x}^1} = \cos \bar{x}^2 \qquad \frac{\partial x^1}{\partial \bar{x}^2} = -\bar{x}^1 \sin \bar{x}^2$$

$$\frac{\partial x^2}{\partial \bar{x}^1} = \sin \bar{x}^2 \qquad \frac{\partial x^2}{\partial \bar{x}^2} = \bar{x}^1 \cos \bar{x}^2$$

$$\bar{g}_{11} = (1 + 4(x^1)^2)(\cos \bar{x}^2)^2 + 2(4x^1x^2)(\cos \bar{x}^2)(\sin \bar{x}^2) + (1 + 4(x^2)^2)(\sin \bar{x}^2)^2$$

$$= (1 + 4(\bar{x}^1 \cos \bar{x}^2)^2)(\cos^2(\bar{x}^2)) + 8(\bar{x}^1 \cos \bar{x}^2)(\bar{x}^1 \sin \bar{x}^2)(\cos \bar{x}^2)(\sin \bar{x}^2) + (1 + 4(\bar{x}^1 \sin \bar{x}^2)^2)(\sin^2(\bar{x}^2))$$

$$= 1 + 4(\bar{x}^1)^2.$$

For homework, find $\bar{g}_{12} = \bar{g}_{21}$, $\bar{g}_{22}$.

Def. Let $(x^1, \dots, x^n)$ and $(\bar{x}^1, \dots, \bar{x}^n)$ be two coordinate systems in a neighborhood of a point $p \in \mathbb{R}^n$. A set of n^{r+s} quantities $T^{i_1,\dots,i_r}_{j_1,\dots,j_s}$ is said to constitute the components of a **tensor type** $(\boldsymbol{r}, \boldsymbol{s})$ if under coordinate transformation these quantities transform according to:

$$\bar{T}^{k_1,\dots,k_r}_{l_1,\dots,l_s} = \sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} \sum_{j_1=1}^{n} \cdots \sum_{j_s=1}^{n} \frac{\partial \bar{x}^{k_1}}{\partial x^{i_1}} \cdots \frac{\partial \bar{x}^{k_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial \bar{x}^{l_1}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{l_s}} T^{i_1,\dots,i_r}_{j_1,\dots,j_s}$$

where all partial derivatives are at p. We define the **rank of the tensor** as $r + s$.

Einstein Summation Notation:

1) In an expression involving the product of 2 tensors we sum from 1 to n over any index that appears as both a subscript and a superscript. Example:

$$A_{ij}B^j \text{ means } \sum_{j=1}^{n} A_{ij}B^j = A_{i1}B^1 + \cdots A_{in}B^n$$

2) In a partial derivative $\frac{\partial \bar{x}^i}{\partial x^j}$, the i is considered a superscript and the j is considered a subscript. Example:

$$\bar{g}_{kl} = \frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^l} g_{ij} \text{ means } \bar{g}_{kl} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^l} g_{ij}$$

3) In a given tensor if a subscript and superscript are the same, then it means the sum from 1 to n over that subscript/superscript. Example:

$$C^{ij}_{kj} \text{ means} \sum_{j=1}^{n} C^{ij}_{kj} = C^{i1}_{k1} + \cdots + C^{in}_{kn} = S^i_k.$$

This is called a contraction of two indices. However, this convention does not apply to sums/differences of tensors. Example:

$A^i + B^i$ or $A^i + B_i$ does not indicate a sum from 1 to n over the index i.

If we contract two indices on an (r, s) tensor, then we get an $(r-1, s-1)$ tensor.

Ex. Show that if we contract on $i = j$ for a $(1, 2)$ tensor with components T^i_{jk}, then the result is a $(0, 1)$ tensor $S_k = T^i_{ik}$.

Since T is a $(1, 2)$ tensor we know (using Einstein notation):

$$\bar{T}^i_{jk} = \frac{\partial \bar{x}^i}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^j} \frac{\partial x^t}{\partial \bar{x}^k} T^r_{st} \ .$$

Setting $i = j$, we get:

$$\bar{S}_k = \bar{T}^i_{ik} = \frac{\partial \bar{x}^i}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial x^t}{\partial \bar{x}^k} T^r_{st} \ .$$

But $\frac{\partial \bar{x}^i}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^i} = \delta^s_r$ (see Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$), so we can write:

$$\bar{S}_k = \bar{T}^i_{ik} = \frac{\partial x^t}{\partial \bar{x}^k} \ \delta^s_r \ T^r_{st} = \frac{\partial x^t}{\partial \bar{x}^k} T^r_{rt} = \frac{\partial x^t}{\partial \bar{x}^k} S_t.$$

So S is a $(0, 1)$ tensor.

Operations on Tensors:

1) If S and T are tensors of type (r, s), then:
   a) $W = S + T$
   b) $B = aS \ ; \ a \in \mathbb{R}$

   are tensors of type (r, s).

2) If S is a tensor of type (r, s) and T is a tensor of type (t, u), then multiplying the components of S with the components of T gives a tensor of type $(r + t, s + u)$.

Ex. Suppose A^i, B^j are components of contravariant vectors (i.e. $(1, 0)$ tensors). Show that $W^{ij} = A^i B^j$ are the components of a $(2, 0)$ tensor.

Since A^i and B^j are contravariant vectors:

$$\bar{A}^i = \sum_{k=1}^{n} \frac{\partial \bar{x}^i}{\partial x^k} A^k \qquad \bar{B}^j = \sum_{l=1}^{n} \frac{\partial \bar{x}^j}{\partial x^l} B^l.$$

Now we have:

$$\bar{W}^{ij} = \bar{A}^i \bar{B}^j = \left(\sum_{k=1}^{n} \frac{\partial \bar{x}^i}{\partial x^k} A^k\right)\left(\sum_{l=1}^{n} \frac{\partial \bar{x}^j}{\partial x^l} B^l\right)$$

$$= \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l} A^k B^l = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l} W^{kl}$$

or in Einstein notation:

$$\bar{W}^{ij} = \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l} W^{kl}$$

So W^{ij} are the components of a $(2, 0)$ tensor.

Theorem: Suppose that T_{ij} are the components of a $(0, 2)$ tensor. If the $n \times n$ matrix (T_{ij}) is invertible on an open set $U \subseteq \mathbb{R}^n$ and the inverse is given by (T^{ij}), then T^{ij} are the components of a $(2, 0)$ tensor.