Higher Order Tensors- HW Problems

1. Let $\overrightarrow{\Phi}(x^1, x^2) = (x^1, x^2, (x^1)^2 + (x^2)^2)$. The metric tensor induced by $\overrightarrow{\Phi}$ is given by

$$g = \begin{pmatrix} 1 + 4(x^1)^2 & 4x^1x^2 \\ 4x^1x^2 & 1 + 4(x^2)^2 \end{pmatrix}.$$

Let $x^1 = \bar{x}^1 cos\bar{x}^2$

$x^2 = \bar{x}^1 sin\bar{x}^2$.

Find the components of the metric tensor $\bar{g}_{12} = \bar{g}_{21}$, $\bar{g}_{22}$ in the $\bar{x}^1, \bar{x}^2$ coordinate system. Make sure your answer is in terms of $\bar{x}^1, \bar{x}^2$.

2. Suppose that B_i are the components of a covariant vector. Show that $\frac{\partial B_j}{\partial x^k} - \frac{\partial B_k}{\partial x^j}$ are the components of a $(0,2)$ tensor.

Hint: The components of a $(0,2)$ tensor must transform like

$$\frac{\partial \bar{B}_j}{\partial \bar{x}^k} - \frac{\partial \bar{B}_k}{\partial \bar{x}^j} = \sum_{\alpha,\beta=1}^{n} \left(\frac{\partial B_\alpha}{\partial x^\beta} - \frac{\partial B_\beta}{\partial x^\alpha}\right) \frac{\partial x^\alpha}{\partial \bar{x}^j} \frac{\partial x^\beta}{\partial \bar{x}^k}.$$

Start with the fact that $\bar{B}_j = \sum_{\alpha=1}^{n} B_\alpha \frac{\partial x^\alpha}{\partial \bar{x}^j}$ and

$\bar{B}_k = \sum_{\gamma=1}^{n} B_\gamma \frac{\partial x^\gamma}{\partial \bar{x}^k}$ and differentiate each of the expressions.

You will also need the chain rule.

3. If (g_{ij}) are the components of a $(0,2)$ tensor and A^i, B^j are each components of $(1,0)$ tensors, show that

$$g_{ij}A^iB^j = \sum_{i,j=1}^{n} g_{ij}A^iB^j$$

is an invariant. That is, it's value does not change when you change coordinate systems. Hint: Recall that $\sum_{i=1}^{n} \frac{\partial \bar{x}^i}{\partial x^j}\frac{\partial x^k}{\partial \bar{x}^i} = \delta_j^k$, the Kronecker delta; see the end of the class notes on functions from $\mathbb{R}^n$ to $\mathbb{R}^m$.

4. Let A_{jk}^i be the components of a $(1,2)$ tensor on $\mathbb{R}^3$. Suppose at a point $x \in \mathbb{R}^3$, $A_{jk}^i = (i + j)k$. So, for example, $A_{23}^1 = (1 + 2)3 = 9$ at x.
   a. Evaluate A_{i2}^i.
   b. Evaluate $A_{3\alpha}^\alpha$.
   c. How many components does the original tensor A have?
   d. How many components does the tensor A have after we contract on $i = j$, i.e. A_{ik}^i?