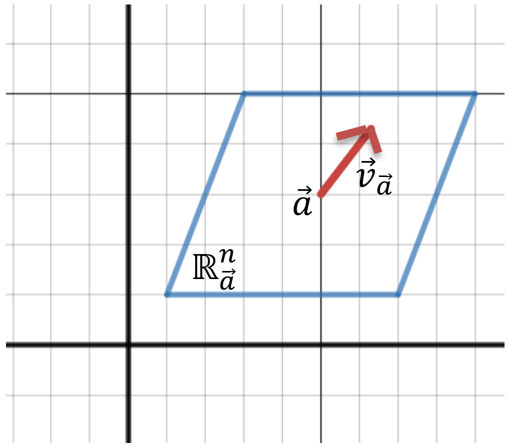
Contravariant and Covariant Vectors

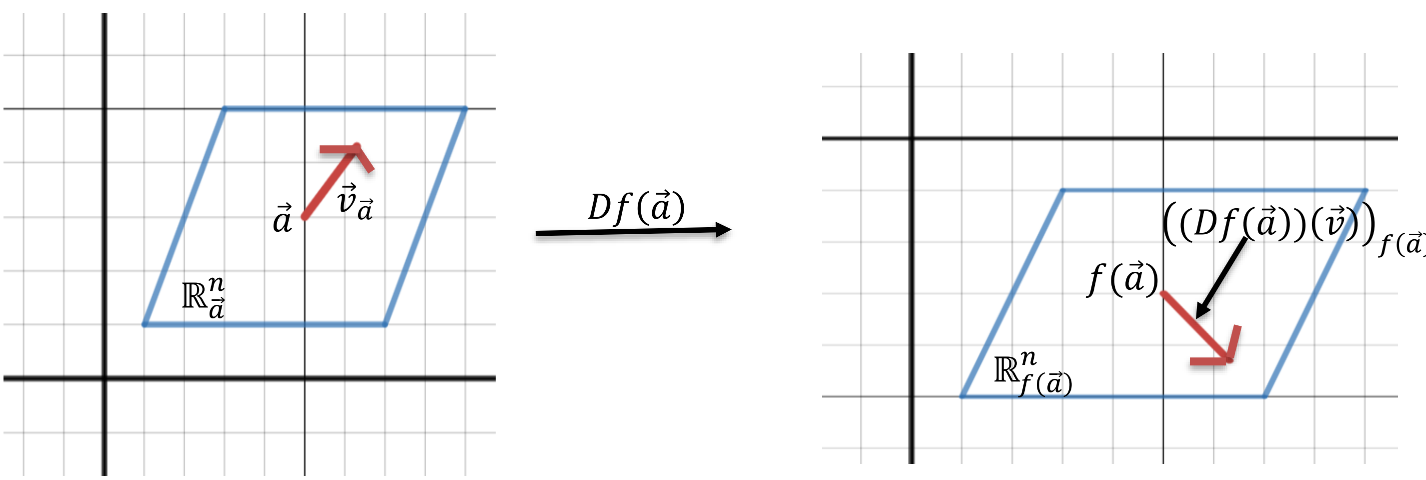
Given any point $\vec{a} \in \mathbb{R}^n$, let $\mathbb{R}^n_{\vec{a}}$ be the set of vectors in $\mathbb{R}^n$ whose "tail" is at $\vec{a} \in \mathbb{R}^n$. That is, $\mathbb{R}^n_{\vec{a}}$ is the set of vectors tangent to $\mathbb{R}^n$ at $\vec{a}$. $\mathbb{R}^n_{\vec{a}}$ is the **tangent space** of $\mathbb{R}^n$ at $\vec{a} \in \mathbb{R}^n$.



If $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable map, then we know $Df(\vec{a})$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. We can interpret this linear transformation as a map from $\mathbb{R}^n_{\vec{a}}$ to $\mathbb{R}^n_{f(\vec{a})}$ by:

$$Df(\vec{a}): \mathbb{R}^n_{\vec{a}} \to \mathbb{R}^n_{f(\vec{a})}$$

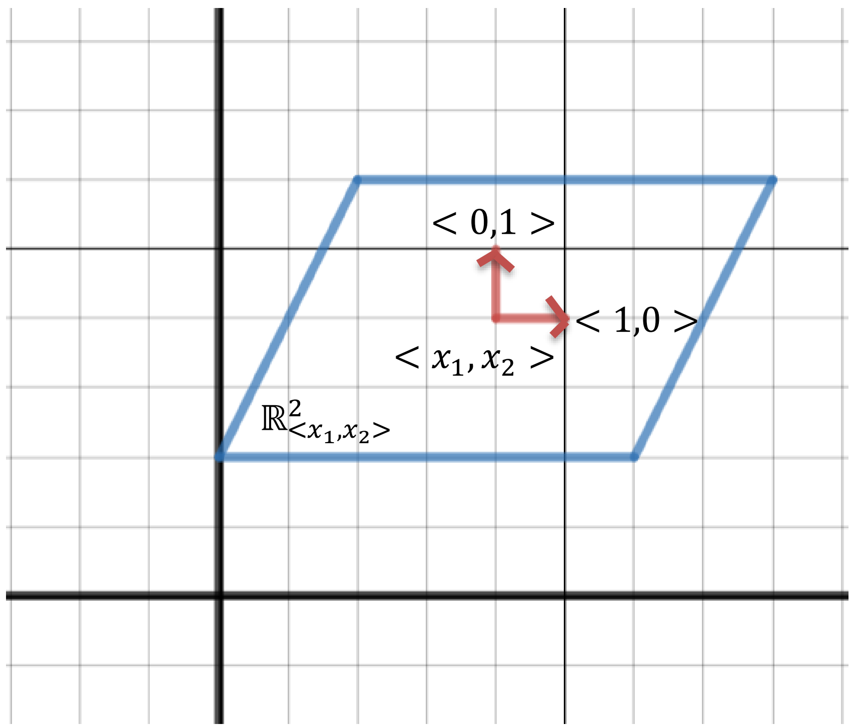
$$\big(Df(\vec{a})\big)(\vec{v}_{\vec{a}}) = \big(Df(\vec{a})(\vec{v})\big)_{f(\vec{a})}.$$



In particular, if $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a change of coordinates, i.e. $\dim\left(\left(Df(\vec{a})\right)\left(\mathbb{R}^n_{\vec{a}}\right)\right) = n$, then $Df(\vec{a})$ is an isomorphism (one-to-one and onto). Thus, $Df(\vec{a})$ maps a basis for $\mathbb{R}^n_{\vec{a}}$ to a basis for $\mathbb{R}^n_{f(\vec{a})}$ for each $\vec{a} \in U \subseteq \mathbb{R}^n$.

If we start with a position vector $\vec{R} = < x_1, x_2 >$ in rectangular coordinates in $\mathbb{R}^2$, then the tangent space at $< x_1, x_2 >$, $\mathbb{R}^2_{<x_1,x_2>}$, is spanned by:

$$\frac{\partial \vec{R}}{\partial x_1} = < 1, 0 > \qquad \frac{\partial \vec{R}}{\partial x_2} = < 0, 1 >$$



In other words, $\{< 1, 0 >, < 0, 1 >\}$ are a basis for the tangent space of $\mathbb{R}^2_{<x_1,x_2>}$ at any point $< x_1, x_2 >$.

Now let's change to polar coordinates (for simplicity let $x_1, x_2 > 0$):

$$r = \bar{x}_1 = \sqrt{x_1^2 + x_2^2} \qquad\qquad \theta = \bar{x}_2 = \tan^{-1}(\tfrac{x_2}{x_1})$$

i.e., $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\quad f(x_1, x_2) = \left(\sqrt{x_1^2 + x_2^2}\,,\ \tan^{-1}(\tfrac{x_2}{x_1})\right).$

As just noted, $Df(x_1, x_2)$ will map a basis for $\mathbb{R}^2_{<x_1,x_2>}$ into a basis for $\mathbb{R}^2_{\vec{f}<x_1,x_2>}$. So under the change of coordinates given by f, how is the new basis for $\mathbb{R}^2_{\vec{f}<x_1,x_2>}$ related to the basis for $\mathbb{R}^2_{<x_1,x_2>}$?

If we call the basis for $\mathbb{R}^2_{\vec{f}<x_1,x_2>}$, $\{\vec{v}_1, \vec{v}_2\}$, then we know that:

$Df(x_1, x_2) < 1{,}0 > = \vec{v}_1$
$Df(x_1, x_2) < 0{,}1 > = \vec{v}_2.$

But we can calculate $Df(x_1, x_2)$:

$$Df(x_1, x_2) = \begin{pmatrix} \frac{\partial \bar{x}_1}{\partial x_1} & \frac{\partial \bar{x}_1}{\partial x_2} \\ \frac{\partial \bar{x}_2}{\partial x_1} & \frac{\partial \bar{x}_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} & \frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ -\frac{x_2}{x_1^2+x_2^2} & \frac{x_1}{x_1^2+x_2^2} \end{pmatrix}$$

$$Df(x_1, x_2) < 1{,}0 >= \begin{pmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} & \frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ -\frac{x_2}{x_1^2+x_2^2} & \frac{x_1}{x_1^2+x_2^2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} \\ -\frac{x_2}{x_1^2+x_2^2} \end{pmatrix} = \vec{v}_1$$

$$Df(x_1, x_2) < 0{,}1 >= \begin{pmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} & \frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ -\frac{x_2}{x_1^2+x_2^2} & \frac{x_1}{x_1^2+x_2^2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ \frac{x_1}{x_1^2+x_2^2} \end{pmatrix} = \vec{v}_2\,.$$

$Df(x_1, x_2)$ maps the basis $\{< 1, 0 >, < 0, 1 >\}$ in $\mathbb{R}^2_{<x_1,x_2>}$ onto the basis

$$\left\{ < \frac{x_1}{\sqrt{x_1^2+x_2^2}},\ -\frac{x_2}{x_1^2+x_2^2} >,\ < \frac{x_2}{\sqrt{x_1^2+x_2^2}},\ \frac{x_1}{x_1^2+x_2^2} > \right\} \text{ in } \mathbb{R}^2_{<\sqrt{x_1^2+x_2^2},\tan^{-1}(\frac{x_2}{x_1})>}.$$

Notice that unlike the basis $\{< 1, 0 >, < 0, 1 >\}$ for $\mathbb{R}^2_{<x_1,x_2>}$, which is the same for any $< x_1, x_2 > \in \mathbb{R}^2$, the new basis for $\mathbb{R}^2_{<\sqrt{x_1^2+x_2^2},\tan^{-1}(\frac{x_2}{x_1})>}$ depends on x_1 and x_2.

If we have a vector in rectangular coordinates $\vec{A} \in \mathbb{R}^2_{<x_1,x_2>}$ given by:

$$\vec{A} =< A^1, A^2 > = A^1 < 1, 0 > \ + A^2 < 0, 1 >$$

how are the components $\bar{A}^1, \bar{A}^2$ of $Df(x_1, x_2)(\vec{A})$, related to A^1, A^2?

That is, what are $\bar{A}^1$ and $\bar{A}^2$ if:

$$Df(x_1, x_2)(< A^1, A^2 >) =< \bar{A}^1, \bar{A}^2 >?$$

Let's use $Df(x_1, x_2) = \begin{pmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} & \frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ -\frac{x_2}{x_1^2+x_2^2} & \frac{x_1}{x_1^2+x_2^2} \end{pmatrix}$ as an example.

Notice, we can rewrite this in matrix form as:

$$\begin{pmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} & \frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ -\frac{x_2}{x_1^2+x_2^2} & \frac{x_1}{x_1^2+x_2^2} \end{pmatrix} \begin{pmatrix} A^1 \\ A^2 \end{pmatrix} = \begin{pmatrix} \bar{A}^1 \\ \bar{A}^2 \end{pmatrix}$$

$$\bar{A}^1 = A^1 \frac{x_1}{\sqrt{x_1^2+x_2^2}} + A^2 \frac{x_2}{\sqrt{x_1^2+x_2^2}}$$

$$\bar{A}^2 = -A^1 \frac{x_2}{x_1^2+x_2^2} + A^2 \frac{x_1}{x_1^2+x_2^2}.$$

In general, for a change of coordinates in $\mathbb{R}^2$, $f: U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, given by:

$$f(x_1, x_2) = (\bar{x}_1(x_1, x_2), \bar{x}_2(x_1, x_2))$$

$$Df(x_1, x_2) = \begin{pmatrix} \frac{\partial \bar{x}_1}{\partial x_1} & \frac{\partial \bar{x}_1}{\partial x_2} \\ \frac{\partial \bar{x}_2}{\partial x_1} & \frac{\partial \bar{x}_2}{\partial x_2} \end{pmatrix}$$

$Df(x_1, x_2)(< A^1, A^2 >) =< \bar{A}^1, \bar{A}^2 >$, becomes

$$\begin{pmatrix} \frac{\partial \bar{x}_1}{\partial x_1} & \frac{\partial \bar{x}_1}{\partial x_2} \\ \frac{\partial \bar{x}_2}{\partial x_1} & \frac{\partial \bar{x}_2}{\partial x_2} \end{pmatrix} \begin{pmatrix} A^1 \\ A^2 \end{pmatrix} = \begin{pmatrix} \bar{A}^1 \\ \bar{A}^2 \end{pmatrix}.$$ Thus:

$$\bar{A}^1 = A^1 \frac{\partial \bar{x}_1}{\partial x_1} + A^2 \frac{\partial \bar{x}_1}{\partial x_2} = \sum_{i=1}^2 A^i \frac{\partial \bar{x}_1}{\partial x_i}$$
$$\bar{A}^2 = A^1 \frac{\partial \bar{x}_2}{\partial x_1} + A^2 \frac{\partial \bar{x}_2}{\partial x_2} = \sum_{i=1}^2 A^i \frac{\partial \bar{x}_2}{\partial x_i}.$$

Def. Let $(x^1, \ldots, x^n)$ and $(\bar{x}^1, \ldots, \bar{x}^n)$ be two coordinate systems in a neighborhood of a point, $p \in \mathbb{R}^n$. An n-tuple $< A^1, \ldots, A^n >$ is said to constitute the components of a **contravariant vector** (or a tensor of type (1,0)) at a point p if the components transform according to the relation:

$$\bar{A}^j = \sum_{i=1}^{n} \frac{\partial \bar{x}^j}{\partial x_i} A^i.$$

So a vector $\vec{A} = < A^1, \ldots, A^n > \in \mathbb{R}_p^n$ is a contravariant vector (that is why we will write coordinates with superscripts from now on).

Another way to see that a vector $\vec{A} = < A^1, \ldots, A^n > \in \mathbb{R}_p^n$ is a contravariant vector is if we represent it by:

$$\vec{A} = \sum_{i=1}^{n} A^i \frac{\partial \vec{R}}{\partial x^i} = \sum_{j=1}^{n} \bar{A}^j \frac{\partial \vec{R}}{\partial \bar{x}^j}$$

Then by the Chain Rule:

$$\frac{\partial \vec{R}}{\partial x^i} = \frac{\partial \vec{R}}{\partial \bar{x}^1} \frac{\partial \bar{x}^1}{\partial x^i} + \cdots + \frac{\partial \vec{R}}{\partial \bar{x}^n} \frac{\partial \bar{x}^n}{\partial x^i} = \sum_{j=1}^{n} \frac{\partial \vec{R}}{\partial \bar{x}^j} \frac{\partial \bar{x}^j}{\partial x^i}$$

$$\vec{A} = \sum_{i=1}^{n} \sum_{j=1}^{n} A^i \frac{\partial \vec{R}}{\partial \bar{x}^j} \frac{\partial \bar{x}^j}{\partial x^i} = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} A^i \frac{\partial \bar{x}^j}{\partial x^i} \right) \frac{\partial \vec{R}}{\partial \bar{x}^j} = \sum_{j=1}^{n} \bar{A}^j \frac{\partial \vec{R}}{\partial \bar{x}^j}$$

$$\boxed{\bar{A}^j = \sum_{i=1}^{n} A^i \frac{\partial \bar{x}^j}{\partial x^i}}$$

Ex. Let $\gamma(t) = \left(x^1(t), \ldots, x^n(t)\right)$ be a curve in $\mathbb{R}^n$. Show that the tangent vector, $\gamma'(t) = \left(\frac{dx^1}{dt}, \ldots, \frac{dx^n}{dt}\right)$, is a contravariant vector.

Let $\vec{A} = \left(\frac{dx^1}{dt}, \ldots, \frac{dx^n}{dt}\right)$ so that the j^{th} component is $A^j = \frac{dx^j}{dt}$.
Let $\bar{\gamma}(t) = \left(\bar{x}^1(t), \ldots, \bar{x}^n(t)\right)$ represent the same curve in another coordinate system $\bar{x}^1, \ldots, \bar{x}^n$ so the tangent vector is:

$$\vec{\bar{A}} = \left(\frac{d\bar{x}^1}{dt}, \ldots, \frac{d\bar{x}^n}{dt}\right) \text{ and } \bar{A}^j = \frac{d\bar{x}^j}{dt}.$$

Using the change of coordinates:

$$\begin{aligned} \bar{x}_1 &= \bar{x}_1(x_1, \ldots, x_n) \\ &\vdots \\ \bar{x}_n &= \bar{x}_n(x_1, \ldots, x_n) \end{aligned}$$

we get by the Chain Rule:

$$\bar{A}^1 = \frac{d\bar{x}^1}{dt} = \frac{\partial \bar{x}^1}{\partial x^1}\frac{dx^1}{dt} + \frac{\partial \bar{x}^1}{\partial x^2}\frac{dx^2}{dt} + \cdots + \frac{\partial \bar{x}^1}{\partial x^n}\frac{\partial x^n}{\partial t}$$

$$= A^1\frac{\partial \bar{x}^1}{\partial x^1} + A^2\frac{\partial \bar{x}^1}{\partial x^2} + \cdots + A^n\frac{\partial \bar{x}^1}{\partial x^n} = \sum_{i=1}^{n} A^i\frac{\partial \bar{x}^1}{\partial x^i}$$

Similarly:

$$\bar{A}^j = \frac{d\bar{x}^j}{dt} = \frac{\partial \bar{x}^j}{\partial x^1}\frac{dx^1}{dt} + \frac{\partial \bar{x}^j}{\partial x^2}\frac{dx^2}{dt} + \cdots + \frac{\partial \bar{x}^j}{\partial x^n}\frac{\partial x^n}{\partial t}$$

$$= A^1\frac{\partial \bar{x}^j}{\partial x^1} + A^2\frac{\partial \bar{x}^j}{\partial x^2} + \cdots + A^n\frac{\partial \bar{x}^j}{\partial x^n} = \sum_{i=1}^{n} A^i\frac{\partial \bar{x}^j}{\partial x^i}$$

So the tangent vector to $\gamma(t)$ is a contravariant vector.

Ex. Let $\gamma(t) = (t\cos t^2, t\sin t^2)$. Given that we know that $\gamma'(t)$ is a contravariant vector, find the components of the tangent vector $(\bar{A}^1, \bar{A}^2)$ in polar coordinates from the components of the tangent vector (A^1, A^2) in rectangular coordinates.

$$\gamma'(t) = (A^1, A^2) = (-2t^2\sin t^2 + \cos t^2, 2t^2\cos t^2 + \sin t^2)$$

$$A^1 = -2t^2\sin t^2 + \cos t^2$$

$$A^2 = 2t^2\cos t^2 + \sin t^2.$$

Since (A^1, A^2) is a contravariant vector:

$$\bar{A}^1 = A^1\frac{\partial\bar{x}^1}{\partial x^1} + A^2\frac{\partial\bar{x}^1}{\partial x^2} \qquad \bar{A}^2 = A^1\frac{\partial\bar{x}^2}{\partial x^1} + A^2\frac{\partial\bar{x}^2}{\partial x^2}$$

$$x^1 = \bar{x}^1\cos\bar{x}^2 = t\cos t^2; \qquad \bar{x}^1 = t$$

$$x^2 = \bar{x}^1\sin\bar{x}^2 = t\sin t^2; \qquad \bar{x}^2 = t^2.$$

Notice that we want $\frac{\partial\bar{x}^i}{\partial x^j}$, but we are given x^1, x^2 in terms of $\bar{x}^1, \bar{x}^2$, and not the other way around. However, remember that if:

$$J = \begin{pmatrix} \frac{\partial x^1}{\partial\bar{x}^1} & \frac{\partial x^1}{\partial\bar{x}^2} \\ \frac{\partial x^2}{\partial\bar{x}^1} & \frac{\partial x^2}{\partial\bar{x}^2} \end{pmatrix}, \text{ then its inverse is } J^{-1} = \begin{pmatrix} \frac{\partial\bar{x}^1}{\partial x^1} & \frac{\partial\bar{x}^1}{\partial x^2} \\ \frac{\partial\bar{x}^2}{\partial x^1} & \frac{\partial\bar{x}^2}{\partial x^2} \end{pmatrix}.$$

So let's calculate J and invert the 2x2 matrix to find J^{-1} and hence the derivatives we want.

$$J = \begin{pmatrix} \cos \bar{x}^2 & -\bar{x}^1 \sin \bar{x}^2 \\ \sin \bar{x}^2 & \bar{x}^1 \cos \bar{x}^2 \end{pmatrix}$$

$$\det(J) = \bar{x}^1 \cos^2 \bar{x}^2 + \bar{x}^1 \sin^2 \bar{x}^2 = \bar{x}^1$$

$$J^{-1} = \frac{1}{\bar{x}^1} \begin{pmatrix} \bar{x}^1 \cos \bar{x}^2 & \bar{x}^1 \sin \bar{x}^2 \\ -\sin \bar{x}^2 & \cos \bar{x}^2 \end{pmatrix} = \begin{pmatrix} \cos \bar{x}^2 & \sin \bar{x}^2 \\ -\frac{\sin \bar{x}^2}{\bar{x}^1} & \frac{\cos \bar{x}^2}{\bar{x}^1} \end{pmatrix}$$

$$\frac{\partial \bar{x}^1}{\partial x^1} = \cos \bar{x}^2 \qquad \frac{\partial \bar{x}^1}{\partial x^2} = \sin \bar{x}^2$$

$$\frac{\partial \bar{x}^2}{\partial x^1} = -\frac{\sin \bar{x}^2}{\bar{x}^1} \qquad \frac{\partial \bar{x}^2}{\partial x^1} = \frac{\cos \bar{x}^2}{\bar{x}^1}\,.$$

So we get:

$$\begin{aligned} \bar{A}^1 &= A^1 \frac{\partial \bar{x}^1}{\partial x^1} + A^2 \frac{\partial \bar{x}^1}{\partial x^2} \\ &= (-2t^2 \sin t^2 + \cos t^2)(\cos \bar{x}^2) + \left((2t^2 \cos t^2 + \sin t^2)(\sin \bar{x}^2)\right) \\ &= (-2t^2 \sin t^2 + \cos t^2)(\cos t^2) + \left((2t^2 \cos t^2 + \sin t^2)(\sin t^2)\right) \\ &= \cos^2 t^2 + \sin^2 t^2 = 1. \end{aligned}$$

$$\begin{aligned} \bar{A}^2 &= A^1 \frac{\partial \bar{x}^2}{\partial x^1} + A^2 \frac{\partial \bar{x}^2}{\partial x^2} \\ &= (-2t^2 \sin t^2 + \cos t^2)\left(-\frac{\sin \bar{x}^2}{\bar{x}^1}\right) + \left((2t^2 \cos t^2 + \sin t^2)\left(\frac{\cos \bar{x}^2}{\bar{x}^1}\right)\right) \\ &= (-2t^2 \sin t^2 + \cos t^2)\left(-\frac{\sin t^2}{t}\right) + \left((2t^2 \cos t^2 + \sin t^2)\left(\frac{\cos t^2}{t}\right)\right) \\ &= 2t \sin^2 t^2 + 2t \cos^2 t^2 = 2t. \end{aligned}$$

Thus we have: $(\bar{A}^1, \bar{A}^2) = (1, 2t)$.

Def. Let $(x^1, .., x^n)$ and $(\bar{x}^1, \dots, \bar{x}^n)$ be two coordinate systems in a neighborhood of a point $p \in \mathbb{R}^n$. An n-tuple $(B_1, \dots, B_n)$ is said to constitute the components of a **covariant vector** (or a $(0,1)$ tensor) at a point, p, if the components transform according to the relation:

$$\bar{B}_j = \sum_{i=1}^{n} \frac{\partial x^i}{\partial \bar{x}^j} B_i \,.$$

Ex. Let $f: \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Show that the gradient $\nabla f = \left(\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^n}\right)$ is a covariant vector.

In this case, $B_j = \frac{\partial f}{\partial x^j}$. if $x^1 = x^1(\bar{x}^1, \dots, \bar{x}^n), \dots, x^n = x^n(\bar{x}^1, \dots, \bar{x}^n)$, then by the Chain Rule in the $\bar{x}^1, \dots, \bar{x}^n$ coordinate system:

$$\nabla f = \left(\frac{\partial f}{\partial \bar{x}^1}, \dots, \frac{\partial f}{\partial \bar{x}^n}\right)$$

$$\bar{B}_j = \frac{\partial f}{\partial \bar{x}^j} = \frac{\partial f}{\partial x^1}\frac{\partial x^1}{\partial \bar{x}^j} + \frac{\partial f}{\partial x^2}\frac{\partial x^2}{\partial \bar{x}^j} + \cdots + \frac{\partial f}{\partial x^n}\frac{\partial x^n}{\partial \bar{x}^j}$$

$$\bar{B}_j = B_1 \frac{\partial x^1}{\partial \bar{x}^j} + B_2 \frac{\partial x^2}{\partial \bar{x}^j} + \cdots + B_n \frac{\partial x^n}{\partial \bar{x}^j}$$

$$= \sum_{i=1}^{n} \frac{\partial x^i}{\partial \bar{x}^j} B_i$$

So ∇f is a covariant vector.

Ex. Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ and we know that

$$\nabla f = (4x^1((x^1)^2 + (x^2)^2) + x^2,\ \ 4x^2((x^1)^2 + (x^2)^2) + x^1).$$

If $x^1 = \bar{x}^1 \cos \bar{x}^2$ and $x^2 = \bar{x}^1 \sin \bar{x}^2$, find ∇f in the $(\bar{x}^1, \bar{x}^2)$ coordinate system. Give the answer only in terms of $\bar{x}^1, \bar{x}^2$ NOT x^1, x^2.

Since ∇f is a covariant vector, if we call $\vec{\bar{B}} = (\bar{B}_1, \bar{B}_2)$ the gradient with respect to $\bar{x}^1, \bar{x}^2$ and $\vec{B} = (B_1, B_2)$ the gradient with respect to x^1, x^2 then we can write:

$$\bar{B}_1 = B_1 \frac{\partial x^1}{\partial \bar{x}^1} + B_2 \frac{\partial x^2}{\partial \bar{x}^1}; \qquad \bar{B}_2 = B_1 \frac{\partial x^1}{\partial \bar{x}^2} + B_2 \frac{\partial x^2}{\partial \bar{x}^2}$$

$$B_1 = 4x^1((x^1)^2 + (x^2)^2) + x^2$$

$$B_2 = 4x^2((x^1)^2 + (x^2)^2) + x^1$$

$$\frac{\partial x^1}{\partial \bar{x}^1} = \cos \bar{x}^2 \qquad \frac{\partial x^2}{\partial \bar{x}^1} = \sin \bar{x}^2$$

$$\frac{\partial x^1}{\partial \bar{x}^2} = -\bar{x}^1 \sin \bar{x}^2 \qquad \frac{\partial x^2}{\partial \bar{x}^2} = \bar{x}^1 \cos \bar{x}^2$$

$$\bar{B}_1 = [4x^1((x^1)^2 + (x^2)^2) + x^2] \cos \bar{x}^2 + [4x^2((x^1)^2 + (x^2)^2) + x^1] \sin \bar{x}^2$$

$$= [4\bar{x}^1 \cos \bar{x}^2\, (\bar{x}^1)^2 + \bar{x}^1 \sin \bar{x}^2] \cos \bar{x}^2$$
$$+ [4\bar{x}^1 \sin \bar{x}^2\, (\bar{x}^1)^2 + \bar{x}^1 \cos \bar{x}^2] \sin \bar{x}^2$$

$$= 4\left(\bar{x}^1\right)^3 (\cos^2(\bar{x}^2) + \sin^2(\bar{x}^2)) + 2\bar{x}^1 (\sin \bar{x}^2)(\cos \bar{x}^2)$$

$$= 4(\bar{x}^1)^3 + 2\bar{x}^1\big((\sin(\bar{x}^2))(\cos(\bar{x}^2)\big).$$

$$\bar{B}_2 = [4x^1((x^1)^2 + (x^2)^2) + x^2](-\bar{x}^1 \sin \bar{x}^2) + [4x^2((x^1)^2 + (x^2)^2) + x^1](\bar{x}^1 \cos \bar{x}^2)$$

$$= [4\bar{x}^1 (\cos \bar{x}^2)(\bar{x}^1)^2 + \bar{x}^1 \sin \bar{x}^2](-\bar{x}^1 \sin \bar{x}^2) + [4\bar{x}^1(\sin \bar{x}^2)(\bar{x}^1)^2 + \bar{x}^1 \cos \bar{x}^2](\bar{x}^1 \cos \bar{x}^2)$$

$$= -(\bar{x}^1)^2(\sin^2(\bar{x}^2)) + (\bar{x}^1)^2(\cos^2(\bar{x}^2)).$$

So in the coordinate system $\bar{x}^1, \bar{x}^2$ the gradient of f is:

$$\nabla f = \left(4(\bar{x}^1)^3 + 2\bar{x}^1\left((\sin(\bar{x}^2))(\cos(\bar{x}^2)\right), \quad \left(\bar{x}^1\right)^2(\cos^2(\bar{x}^2) - \sin^2(\bar{x}^2))\right).$$