Geodesics

Def. Let M be a smooth manifold and $\gamma: I \to M$ a smooth curve. γ is called a **geodesic** if $\nabla_{\gamma'}(\gamma') = 0$ ($\nabla_{\gamma'}(\gamma')$ is called the **acceleration vector field** of γ).

Note: This definition depends on the parametrization of the curve.

The components of $D_t(V)$ are given by:

$$\left(\dot{v}^k + \sum_{i,j=1}^{n} \Gamma_{ij}^k \left(\dot{\gamma}^i\right)\left(v^j\right)\right); \qquad \text{where } V(t) = (v^1(t), \dots, v^n(t)) .$$

Thus if $V(t) = (\dot{\gamma}^1(t), \dots, \dot{\gamma}^n)$, the components of $D_t(V)$ are given by:

$$\frac{d^2\gamma^k}{dt^2} + \sum_{i,j=1}^{n} \Gamma_{ij}^k \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} .$$

So to find a geodesic on a manifold M we have to solve the system of differential equations that come from $D_t(\gamma') = 0$; that is:

$$\frac{d^2\gamma^k}{dt^2} + \sum_{i,j=1}^{n} \Gamma_{ij}^k \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0; \qquad k = 1, \dots, n.$$

Ex. Let H be the upper half plane, $H = \{(x, y) |\ y > 0\}$, and let the metric on H be given as $g = \frac{1}{y^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. What are the geodesics in H?

From direct calculation (and an earlier HW problem) we know:

$$\Gamma_{12}^1 = \Gamma_{21}^1 = -\frac{1}{y}, \quad \Gamma_{11}^2 = \frac{1}{y}, \qquad \Gamma_{22}^2 = -\frac{1}{y}, \text{ all other } \Gamma_{ij}^k = 0.$$

Thus the differential equations for the geodesics are:

$$k = 1 \quad \frac{d^2\gamma^1}{dt^2} + \sum_{i,j=1}^n \Gamma_{ij}^1 \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0.$$

The only nonzero Γ_{ij}^1 are $\Gamma_{12}^1 = \Gamma_{21}^1 = -\frac{1}{y} = -\frac{1}{\gamma^2}$, since $\gamma(t) = (x(t), y(t))$. Thus we have:

$$\frac{d^2\gamma^1}{dt^2} + \Gamma_{12}^1 \frac{d\gamma^1}{dt}\frac{d\gamma^2}{dt} + \Gamma_{21}^1 \frac{d\gamma^2}{dt}\frac{d\gamma^1}{dt} = 0$$

$$\frac{d^2\gamma^1}{dt^2} - \frac{2}{\gamma^2}\frac{d\gamma^1}{dt}\frac{d\gamma^2}{dt} = 0; \quad \text{or}$$

$$\gamma^2 \frac{d^2\gamma^1}{dt^2} = 2\frac{d\gamma^1}{dt}\frac{d\gamma^2}{dt}.$$

$k = 2 \qquad \frac{d^2\gamma^2}{dt^2} + \sum_{i,j=1}^{n} \Gamma_{ij}^2 \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0.$

The only nonzero Γ_{ij}^2 are: $\Gamma_{11}^2 = \frac{1}{y} = \frac{1}{\gamma^2}; \quad \Gamma_{22}^2 = -\frac{1}{y} = -\frac{1}{\gamma^2}.$

Thus we have:

$$\frac{d^2\gamma^2}{dt^2} + \Gamma_{11}^2 \frac{d\gamma^1}{dt}\frac{d\gamma^1}{dt} + \Gamma_{22}^2 \frac{d\gamma^2}{dt}\frac{d\gamma^2}{dt} = 0$$

$$\frac{d^2\gamma^2}{dt^2} + \frac{1}{\gamma^2}\left(\frac{d\gamma^1}{dt}\right)^2 - \frac{1}{\gamma^2}\left(\frac{d\gamma^2}{dt}\right)^2 = 0 \; ; \quad \text{or}$$

$$\gamma^2 \frac{d^2\gamma^2}{dt^2} = \left(\frac{d\gamma^2}{dt}\right)^2 - \left(\frac{d\gamma^1}{dt}\right)^2.$$

So the system of differential equations for the geodesics are:

$$\gamma^2 \frac{d^2\gamma^1}{dt^2} = 2\frac{d\gamma^1}{dt}\frac{d\gamma^2}{dt}$$

$$\gamma^2 \frac{d^2\gamma^2}{dt^2} = \left(\frac{d\gamma^2}{dt}\right)^2 - \left(\frac{d\gamma^1}{dt}\right)^2.$$

Notice that for $c_1, c_2, c_3 \in \mathbb{R}$

$$\gamma^1(t) = c_1$$

$$\gamma^2(t) = c_2 e^{c_3 t}$$

are solutions to the system of equations (vertical lines in H).

In addition we have the following solutions for $r, k \in \mathbb{R}$:

$$\gamma^1(t) = -(r)\tanh(t) + k = -r\left(\frac{e^t - e^{-t}}{e^t + e^{-t}}\right) + k$$

$$\gamma^2(t) = (r)\operatorname{sech}(t) = \frac{2r}{e^t + e^{-t}}$$

This second set of solutions represent circles of radius r centered at $(k, 0)$ since:

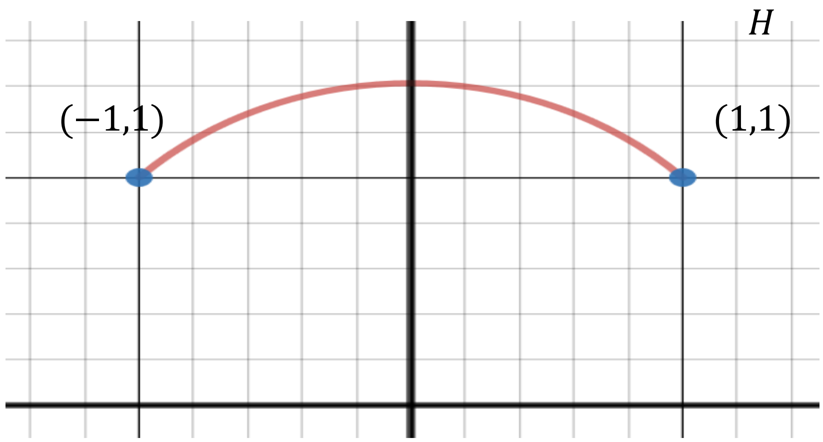
$$(\gamma^1(t))^2 + (\gamma^2(t) - k)^2 = r^2 \tanh^2(t) + r^2 \operatorname{sech}^2(t)$$

$$= r^2 [\frac{\sinh^2(t)}{\cosh^2(t)} + \frac{1}{\cosh^2(t)}]$$

$$= r^2 \left[\frac{\sinh^2(t) + 1}{\cosh^2(t)}\right] = r^2.$$

These two sets of solutions represent all of the solutions to the system of differential equations. Thus all of the geodesics in H are either semi-circles of radius r centered on the x-axis or vertical line segments.

Ex. Find the length of the geodesic from $(1,1)$ to $(-1,1)$ in H.



Since the points are symmetric about the y-axis, the center of the circle (the geodesic) must be $(0,0)$ and the radius is $\sqrt{2}$.

Thus we have:

$$\gamma^1(t) = -(\sqrt{2})\tanh(t) = -\sqrt{2}\left(\frac{e^t - e^{-t}}{e^t + e^{-t}}\right) = 1$$

$$\gamma^2(t) = (2)\,\text{sech}(t) = \frac{2\sqrt{2}}{e^t + e^{-t}} = 1.$$

Solving simultaneously we get: $t = \frac{1}{2}\ln\left(3 - 2\sqrt{2}\right).$

Similarly, at $(-1,1)$ we have:

$$\gamma^1(t) = -(\sqrt{2})\tanh(t) = -\sqrt{2}\left(\frac{e^t - e^{-t}}{e^t + e^{-t}}\right) = -1$$

$$\gamma^2(t) = (\sqrt{2})\,\mathrm{sech}(t) = \frac{2\sqrt{2}}{e^t+e^{-t}} = 1.$$

Solving simultaneously we get: $t = \frac{1}{2}\ln(3 + 2\sqrt{2})$.

$$L = \int_{t=\frac{1}{2}\ln(3-2\sqrt{2})}^{t=\frac{1}{2}\ln(3+2\sqrt{2})} (\frac{1}{y^2}(g_{11}(\dot{\gamma}^1)^2 + 2g_{12}(\dot{\gamma}^1)(\dot{\gamma}^2) + g_{22}(\dot{\gamma}^2)^2)^{\frac{1}{2}}\,dt$$

$$= \int_{t=\frac{1}{2}\ln(3-2\sqrt{2})}^{t=\frac{1}{2}\ln(3+2\sqrt{2})} [\frac{1}{2\mathrm{sech}^2(t)}(2\,\mathrm{sech}^4\, t + 2\,(\mathrm{sech}^2\, t)(\tanh^2\, t)]^{\frac{1}{2}}\,dt$$

$$= \int_{t=\frac{1}{2}\ln(3-2\sqrt{2})}^{t=\frac{1}{2}\ln(3+2\sqrt{2})} (\mathrm{sech}^2\, t + \tanh^2\, t)^{\frac{1}{2}}\,dt = \int_{t=\frac{1}{2}\ln(3-2\sqrt{2})}^{t=\frac{1}{2}\ln(3+2\sqrt{2})} 1\,dt$$

$$= \frac{1}{2}\ln(3 + 2\sqrt{2}) - \frac{1}{2}\ln(3 - 2\sqrt{2}) = \frac{1}{2}\ln\,(\frac{(3+2\sqrt{2})}{(3-2\sqrt{2})}) \approx 1.76.$$

Notice that a line segment between the two points,

$\gamma^1(t) = t,$

$\gamma^2(t) = 1; \; -1 \leq t \leq 1$

has length equal to 2 (i.e. longer than the geodesic):

$$L = \int_{t=-1}^{t=1} (\frac{1}{y^2} (g_{11}(\dot{\gamma}^1)^2 + 2g_{12}(\dot{\gamma}^1)(\dot{\gamma}^2) + g_{22}(\dot{\gamma}^2)^2)^{\frac{1}{2}} \, dt$$

$$= \int_{t=-1}^{t=1} 1dt = 2$$