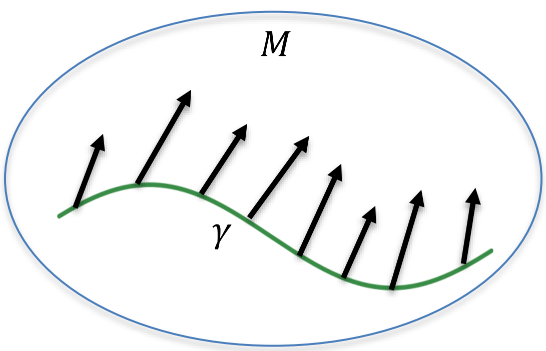
Vector Fields Along Curves

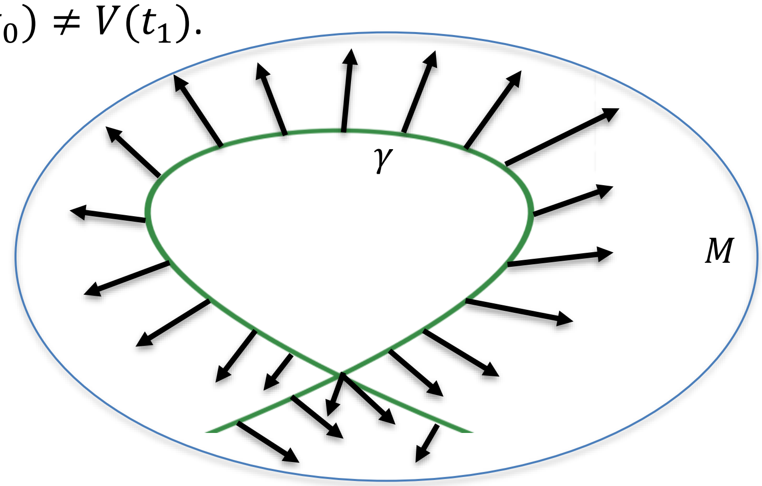
Given a curve, $\gamma(t)$, on a smooth manifold M and a vector $V_0 \in T_{\gamma(t_0)}M$ we want to be able to say what it means to transport V_0 in a "parallel" fashion along the curve $\gamma(t)$ to a vector $V_1 \in T_{\gamma(t_1)}M$. This notion of "parallel transport" will become important when we discuss the Riemann curvature tensor on M. it's also important in the discussion of geodesic curves on a manifold.

Def. Let M be a smooth manifold and $\gamma: I \to M$ be a smooth curve in M where I is an interval in $\mathbb{R}$. We call V a **vector field along $\boldsymbol{\gamma}$** if for each $t \in I$, $V(t) \in T_{\gamma(t)}M$ and V defines a smooth map $I \to TM$. We denote the set of all smooth vector fields on M along γ by $\chi_\gamma(M)$.

A vector field along a curve is not necessarily the restriction of a vector field on M to γ. For example, whenever a curve self-intersects:
$\gamma(t_0) = \gamma(t_1)$ with $t_0 \neq t_1$, but $V(t_0) \neq V(t_1)$.



Vector field along γ that is the restriction of a vector field on M.



Vector fields along γ that is not the restriction of a vector field on M.

We defined a connection on Mas a map, $\nabla: \chi(M) \times \chi(M) \to \chi(M)$.
Now we want to define a map $D_t: \chi_\gamma(M) \to \chi_\gamma(M)$.

Def. Let M be a smooth manifold with a connection ∇ and $\gamma: I \to M$ a smooth curve on M, then the unique map $D_t: \chi_\gamma(M) \to \chi_\gamma(M)$ such that:

1) $D_t(V + W) = D_t(V) + D_t(W)$
2) $D_t(fV) = (\frac{df}{dt})V + (f)\big(D_t(V)\big)$
3) If V extends to a vector field $Y \in \chi(M)$, then $D_t(V) = \nabla_{\gamma'(t)}(Y)$

is called the **covariant derivative along γ**.

If D_t exists, we can use properties 1, 2, and 3 to find a formula for it. Let U be a coordinate patch on M with coordinates $(x^1, \dots, x^n)$. For any $V \in \chi_\gamma(M)$ we can write:

$$V = v^i \partial_i = v^i \frac{\partial}{\partial x^i}; \qquad v^i \in C^\infty(I).$$

By conditions 1 and 2:

$$D_t(V) = D_t\big(v^i \partial_i\big) = \dot{v}^i \partial_i + v^i D_t(\partial_i) \quad \text{where} \quad \dot{v}^i = \frac{dv^i}{dt}.$$

If we write $\gamma(t) = \big(\gamma^1(t), \gamma^2(t), \dots, \gamma^n(t)\big)$, i.e., $x^j = \gamma^j(t)$, then:

$$\gamma'(t) = \sum_{j=1}^{n} \dot{\gamma}^j \, \partial_j .$$

By condition 3:

$$D_t\big(\partial_j\big) = \nabla_{\gamma'(t)}\big(\partial_j\big) = \sum_{i=1}^{n} (\dot{\gamma}^i) \, \nabla_{\partial_i}\big(\partial_j\big) = \dot{\gamma}^i\big(\Gamma_{ij}^k\big)(\partial_k).$$

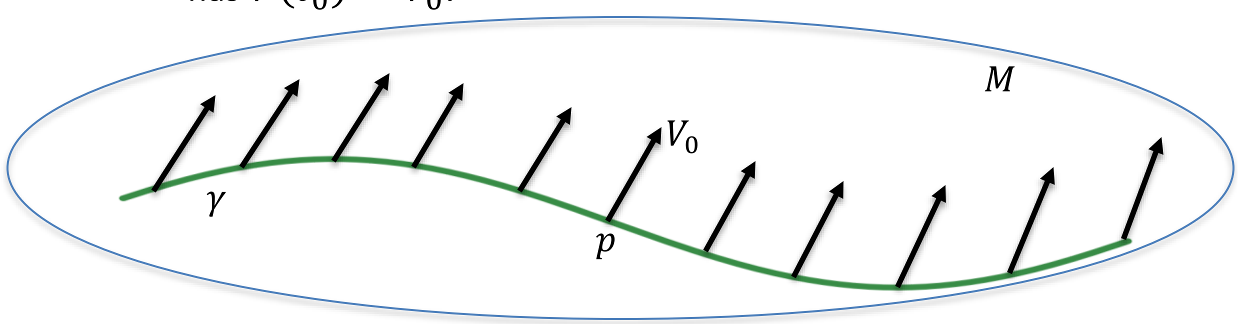
Thus we have:

$$D_t(V) = (\dot{v}^j)\partial_j + \left(\Gamma_{ij}^k(\dot{\gamma}^i)(v^j)\partial_k\right) = \left(\dot{v}^k + \Gamma_{ij}^k(\dot{\gamma}^i)(v^j)\right)\partial_k \,.$$

To show that D_t exists, one can start with this formula and show it satisfies conditions 1, 2, and 3.

Def. Let M be a smooth manifold with a connection ∇ and let $\gamma: I \to M$ be a smooth curve on M. A vector field V along γ is called **parallel** if $D_t(V) = 0$ for all $t \in I$.

Proposition: Let M be a smooth manifold with a connection ∇ and let $\gamma: I \to M$ be a smooth curve on M, where I is a compact (i.e. closed and bounded) interval of $\mathbb{R}$.
Let $t_0 \in I$, set $p = \gamma(t_0)$, and let V_0 be any vector in T_pM.
There exists a unique vector field of M along γ that is parallel and has $V(t_0) = V_0$.



In this case, we are parallel transporting the vector V_0 along γ. That is, $V(t)$ is the parallel transport of V_0 along γ. The existence and uniqueness of this vector field along γ comes from the existence of a unique solution to a system of differential equations. Specifically, if $V(t) = \left(v^1(t), \dots, v^n(t)\right)$, then we need to show there is a unique $V(t)$ such that:

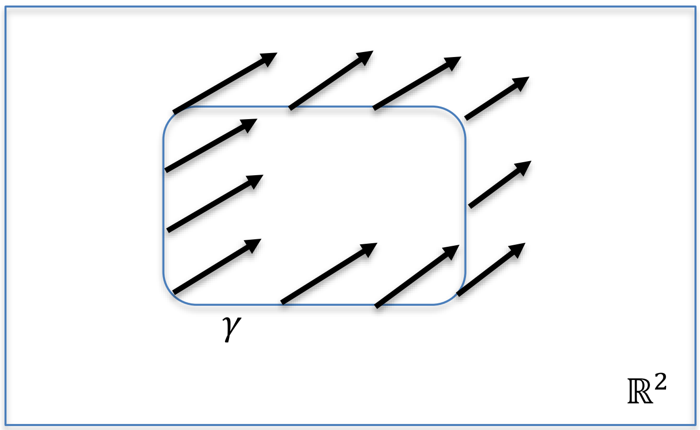
$\dot{v}^k + \Gamma_{ij}^k \dot{\gamma}^i\, v^j = 0$ for $k = 1, \dots, n$ with $V(t_0) = \left(v^1(t_0), \dots, v^n(t_0)\right)$

This comes from a theorem in differential equations.

Ex. Let $M = \mathbb{R}^2$ with the standard metric. Then $\Gamma_{ij}^k = 0$ and if $V_0 = (v_0^1, v_0^2)$ our parallel vector field $V(t)$ must satisfy:

$$D_t(V^j) = 0 = \frac{dv^j}{dt}; \quad j = 1,2.$$

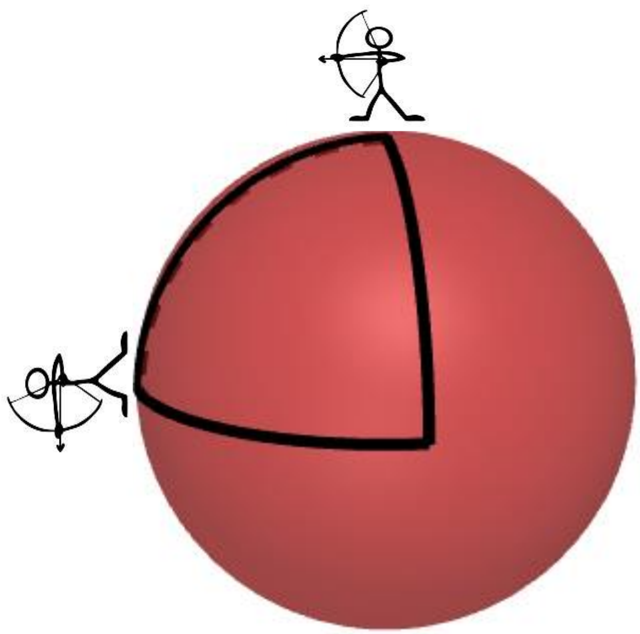
Thus we have: $V(t) = \left(v^1(t), v^2(t)\right) = (v_0^1, v_0^2)$.



Since the components of $V(t)$ are constant, if we parallel transport any vector around any closed curve γ on M we get back to the same vector. For general manifolds this does not happen, in fact, the failure to return to the same vector is a measure of curvature.

Saying $V(t)$ is a parallel vector field along γ means that $D_t(V)$ only has components in normal directions to the tangent space $T_{\gamma(t)}M$. That is, if we project $D_t(V)$ on to $T_{\gamma(t)}M$ we get the zero vector. Thus $V(t)$ is changing just as $T_{\gamma(t)}M$ is changing as t changes.

Suppose a person is standing at the north pole of a sphere (the point $(0,0,1)$) with a bow and arrow pointing parallel to the x-axis – we'll think of the arrow as the vector we are parallel transporting (the vector $< -1,0,0 >$). If the person now walks, without turning, along the line of longitude $\theta = \pi$ towards the equator (the point $(-1,0,0)$), then the arrow will be pointing down, that is, parallel to the z-axis (the vector $< 0,0,-1 >$) at the point $(-1,0,0)$.



If the person does not turn and walks (sideways) $\frac{1}{4}$ of the way around the sphere to the point $(0,-1,0)$, their arrow will continue to point straight down, parallel to the z-axis.

Now the person walks back to the north pole (without turning, thus they are walking backwards) along the line of longitude $\theta =$. When they get to the north pole, the arrow is now parallel to the y-axis (the vector $< 0,-1,0 >$), which is different from the beginning when the arrow was parallel to the x-axis (the vector $< -1,0,0 >$).

Ex. Let $S^2 \subseteq \mathbb{R}^3$ be the unit sphere where:

$$\vec{\Phi}(\theta,\phi) = (\cos\theta\sin\phi\,,\sin\theta\sin\phi\,,\cos\phi)$$
$$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi.$$

Parallel transport the vector $< 0,1 >$ at the point $\left(0,\frac{\pi}{2}\right)$ to the north pole along the following two paths and show that they are different vectors.

1) Path 1: along $\theta = 0, \quad \phi = \frac{\pi}{2} - t, \quad 0 \le t \le \frac{\pi}{2}$

2) Path 2: along $\theta = t, \quad \phi = \frac{\pi}{2}, \quad 0 \le t \le \frac{\pi}{2}$

and then along $\theta = \frac{\pi}{2}, \quad \phi = \frac{\pi}{2} - t, \quad 0 \le t \le \frac{\pi}{2}$.

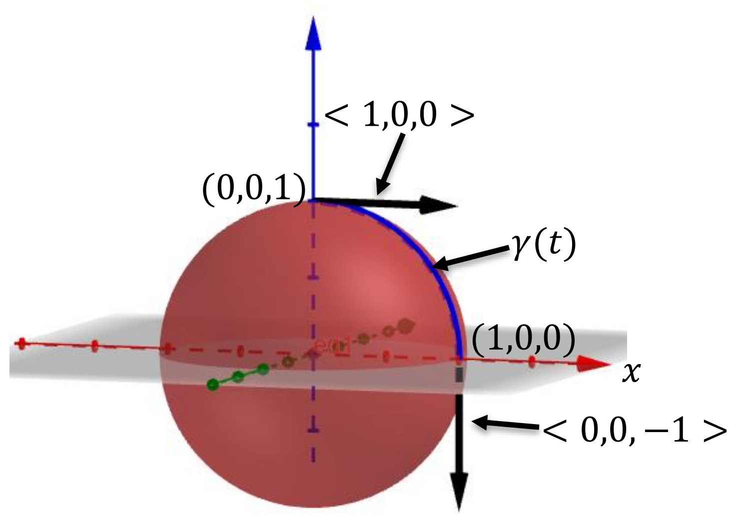
1) $\gamma(t) = \left(0,\frac{\pi}{2} - t\right)$; $0 \le t \le \frac{\pi}{2}$

Notice that $\gamma(0) = \left(0,\frac{\pi}{2}\right)$, which is $(1,0,0)$ in $\mathbb{R}^3$ and $\gamma\left(\frac{\pi}{2}\right) = (0,0)$, which is $(0,0,1)$ in $\mathbb{R}^3$. We have seen that for $\vec{\Phi}$ we have:

$$\vec{\Phi}_\theta = < -\sin\theta\sin\phi\,,\cos\theta\sin\phi\,,0 >$$

$$\vec{\Phi}_\phi = < \cos\theta\cos\phi\,,\sin\theta\cos\phi\,,-\sin\phi >$$

So the vector $< 0,1 > = \vec{\Phi}_\phi$ at $\gamma(0) = \left(0,\frac{\pi}{2}\right)$ is $\vec{\Phi}_\phi\left(0,\frac{\pi}{2}\right) = < 0,0,-1 >$ in $\mathbb{R}^3$.



We also know:

$$g = \begin{pmatrix} \sin^2\phi & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Gamma_{21}^1 = \Gamma_{12}^1 = \cot\phi$$

$$\Gamma_{11}^2 = -\sin\phi\cos\phi$$

$$\Gamma_{ij}^k = 0 \text{ otherwise.}$$

To parallel transport $< 0, 1 >$ we solve $D_t V = \dot{v}^k + \Gamma_{ij}^k(\dot{\gamma}^i)v^j = 0$ where $V(t) = < v^1(t), v^2(t) >; \quad V(0) = < v^1(0), v^2(0) > = < 0, 1 >$.

$$\gamma(t) = (\gamma^1(t), \gamma^2(t)) = \left(0, \frac{\pi}{2} - t\right)$$

$$\gamma'(t) = (\dot{\gamma}^1(t), \dot{\gamma}^2(t)) = (0, -1).$$

$k = 1$: $$\dot{v}^1 + \Gamma_{ij}^1(\dot{\gamma}^i)v^j = 0$$

$$\dot{\gamma}^1(t) = 0; \quad \dot{\gamma}^2(t) = -1,$$

so $$\dot{v}^1 + \Gamma_{2j}^1(-1)v^j = 0.$$

$$\Gamma_{21}^1 = \cot\phi; \quad \Gamma_{22}^1 = 0,$$

thus we have $$\dot{v}^1 - (\cot\phi)v^1 = 0.$$

$$\gamma(t) = \left(0, \frac{\pi}{2} - t\right) = (\theta, \phi)$$

since $$\frac{\pi}{2} - t = \phi$$ so,

$$\dot{v}^1 - \left(\cot\left(\frac{\pi}{2} - t\right)\right)v^1 = 0$$

$$\dot{v}^1 - (\tan t)v^1 = 0.$$

$$\dot{v}^1 = (\tan t)v^1$$

$$\frac{\dot{v}^1}{v^1} = \tan t$$

$$\int \frac{\dot{v}^1}{v^1} dt = \int (\tan t)\, dt = \int \frac{\sin t}{\cos t} dt$$

$$\ln v^1 + c_1 = -\ln(\cos t) + c_2$$

$$\ln v^1 = -\ln(\cos t) + c_3$$

$$v^1 = e^{-\ln(\cos t) + c_3} = c_4 \sec t.$$

$$v^1(0) = c_4(1) = 0 \quad \Rightarrow \quad c_4 = 0 \quad \Rightarrow \quad v^1(t) = 0.$$

$k = 2$: $$\dot{v}^2 + \Gamma_{ij}^2(\dot{\gamma}^i)v^j = 0$$

$$\dot{\gamma}^1(t) = 0; \quad \dot{\gamma}^2(t) = -1.$$

$$\Rightarrow \qquad \dot{v}^2 - \Gamma_{2j}^2 v^j = 0.$$

$$\Gamma_{21}^2 = \Gamma_{22}^2 = 0$$

$$\Rightarrow \qquad \dot{v}^2 = 0.$$

$$\Rightarrow \qquad v^2 = c.$$

$$v^2(0) = 1$$

$$\Rightarrow \qquad v^2(t) = 1$$

$$v^1(t) = 0\,.$$

$$\Rightarrow \qquad V(t) = <0, 1> = \overrightarrow{\Phi}_\phi(t).$$

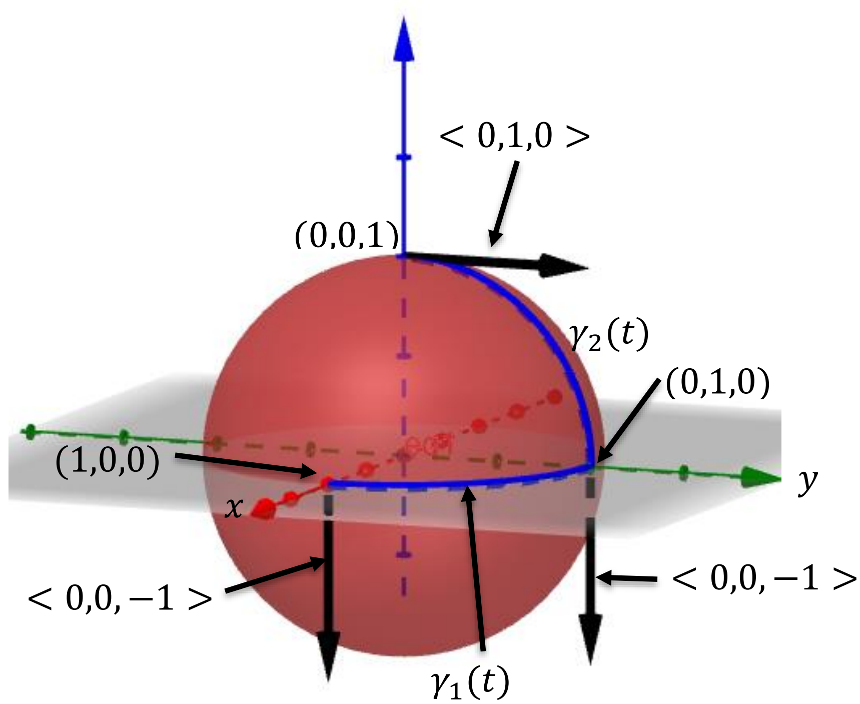
At the north pole $\theta = 0, \ \ \phi = 0$, and:

$$\overrightarrow{\Phi}_{\phi}(0,0) = <(\cos 0)(\cos 0), (\sin 0)(\sin 0), -\sin 0> = <1,0,0>_{\mathbb{R}^3}$$

So the parallel transport of $<0,1>$ at the point $\theta = 0, \ \ \phi = \frac{\pi}{2}$ (which in $\mathbb{R}^3$ is $(1,0,0)$ and $<0,1> = <0,0,-1>$) to the north pole ($\theta = 0, \ \ \phi = 0$) is again the vector $<0,1>$. However, at $\theta = 0, \ \ \phi = 0$:

$$<0,1> = \overrightarrow{\Phi}_{\phi}(0,0) = <1,0,0> \in \mathbb{R}^3.$$

2) First parallel transport $<0,1>$ at $\theta = 0, \ \phi = \frac{\pi}{2}$ to $\theta = \frac{\pi}{2}, \ \phi = \frac{\pi}{2}$ along $\gamma_1(t) = \left(t, \frac{\pi}{2}\right); \quad 0 \le t \le \frac{\pi}{2}$



We must solve the differential equations:

$$D_t V = \dot{v}^k + \Gamma_{ij}^k(\dot{\gamma}_1^i)(v^j) = 0 \quad \text{where } \dot{\gamma}_1(t) = (1,0).$$

$k = 1$: $$\dot{v}^1 + \Gamma_{ij}^1(\dot{\gamma}_1^i)(v^j) = 0.$$

$$\dot{\gamma}_1^1(t) = 1; \qquad \dot{\gamma}_1^2(t) = 0$$

$$\Rightarrow \qquad \dot{v}^1 + \Gamma_{1j}^1(v^j) = 0.$$

$$\Gamma_{11}^1 = 0; \qquad \Gamma_{12}^1 = \cot\phi$$

$$\Rightarrow \qquad \dot{v}^1 + (\cot\phi)v^2 = 0.$$

But $\phi = \frac{\pi}{2}$ so $\cot\phi = 0$, and thus we get $\dot{v}^1 = 0 \Rightarrow v^1 =$ constant, and since $v^1(0) = 0$, $\Longrightarrow$ $v^1(t) = 0$.

$k = 2$: $$\dot{v}^2 + \Gamma_{ij}^2(\dot{\gamma}_1^i)(v^j) = 0.$$

$$\dot{\gamma}_1^1(t) = 1; \qquad \dot{\gamma}_1^2(t) = 0$$

$$\Rightarrow \qquad \dot{v}^2 + \Gamma_{1j}^2(\dot{\gamma}_1^1)(v^j) = 0.$$

$$\Gamma_{11}^2 = -(\sin\phi)\cos\phi\,; \qquad \Gamma_{12}^2 = 0$$

But for $\phi = \frac{\pi}{2}$, $\cos\phi = 0$ so $\Gamma_{11}^2 = 0$, and $\dot{v}^2 = 0 \Rightarrow v^2(t) =$ constant, and since $v^2(0) = 1$, $\Longrightarrow$ $v^2(t) = 1$.

$V(t) = < 0,1 >$ along $\gamma_1(t) = \left(t, \frac{\pi}{2}\right)$.

at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$: $< 0,1 > = \overrightarrow{\Phi}_\phi\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = < 0,0,-1 > \in \mathbb{R}^3$.

So the parallel transport of $< 0, 1 >$ at $\left(0, \frac{\pi}{2}\right)$ (which is $< 0, 0, -1 >$ at $(1, 0, 0)$ in $\mathbb{R}^3$) to $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ along $\gamma_1(t)$ is the vector:

$$< 0, 1 > = < 0, 0, -1 >_{\mathbb{R}^3} \text{ at } \left(\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Now we need to parallel transport $< 0, 1 >$ along the curve:

$$\gamma_2(t) = \left(\frac{\pi}{2}, \frac{\pi}{2} - t\right); \quad 0 \le t \le \frac{\pi}{2}$$

from the point $\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = (0, 1, 0) \in \mathbb{R}^3$ to $\left(\frac{\pi}{2}, 0\right) = (0, 0, 1) \in \mathbb{R}^3$.

$$\dot{\gamma}_2 = (0, -1) \ ; \quad \dot{\gamma}_2^1 = 0 \ ; \quad \dot{\gamma}_2^2 = -1.$$

$$D_t V = \dot{v}^k + \Gamma_{ij}^k \left(\dot{\gamma}_2^i\right)\left(v^j\right) = 0.$$

$k = 1$:

$$\dot{v}^1 + \Gamma_{ij}^1 \left(\dot{\gamma}_2^i\right)\left(v^j\right) = 0$$

$$\dot{v}^1 + \Gamma_{2j}^1 (-1)\left(v^j\right) = 0.$$

$$\Gamma_{21}^1 = \cot\phi \, ; \qquad \Gamma_{22}^1 = 0$$

$$\Rightarrow \qquad \dot{v}^1 - (\cot\phi) v^1 = 0.$$

$$\phi = \frac{\pi}{2} - t$$

$$\Rightarrow \qquad \dot{v}^1 - \left(\cot\left(\frac{\pi}{2} - t\right)\right) v^1 = 0$$

$$\dot{v}^1 - (\tan t) v^1 = 0.$$

$$\Rightarrow \qquad \dot{v}^1 = (\tan t) v^1;$$

$$\frac{\dot{v}^1}{v^1} = \frac{sint}{cost}$$

$$\int \frac{\dot{v}^1}{v^1}\,dt = \int \frac{\sin t}{\cos t}\,dt$$

$$\ln v^1 = -\ln\cos t + c$$

$$v^1 = e^{-\ln(\cos t)+c} = c_2 \sec t.$$

$$0 = v^1(0) = c_2(\sec 0) = c_2; \quad \Longrightarrow \quad v^1(t) = 0.$$

$k = 2$: $\dot{v}^2 + \Gamma^2_{ij}(\dot{\gamma}^i_2)(v^j) = 0$

$$\dot{\gamma}^1_2 = 0; \qquad \dot{\gamma}^2_2 = -1$$

$$\Rightarrow \qquad \dot{v}^2 + \Gamma^2_{2j}(-1)(v^j) = 0.$$

$$\Gamma^2_{21} = 0 = \Gamma^2_{22}$$

$\Rightarrow$ $\dot{v}^2 = 0$ and $v^2(t) =$ constant.

$1 = v^2(0) =$ constant

$$v^2(t) = 1.$$

$$V(t) = <0,1>.$$

Thus $<0,1>$ at the point $\left(\frac{\pi}{2},\frac{\pi}{2}\right) = (0,1,0)$ is parallel transported to $<0,1>$ at the point $\left(\frac{\pi}{2},0\right) = (0,0,1)$.

However, at $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$:

$$<0,1> = \overrightarrow{\Phi}_\phi\left(\frac{\pi}{2},\frac{\pi}{2}\right)$$

$$= <0,0,-1>_{\mathbb{R}^3}.$$

But at $\left(\frac{\pi}{2}, 0\right)$:

$$< 0, 1 > = \vec{\Phi}_\phi\left(\frac{\pi}{2}, 0\right) < \left(\cos\frac{\pi}{2}\right)(\cos 0), \left(\sin\frac{\pi}{2}\right)\ (\cos 0), -\sin 0 >$$

$$= < 0, 1, 0 >_{\mathbb{R}^3}\ .$$

Thus, along γ_1 and then γ_2, the vector $< 0, 1 > = < 0, 0, -1 >_{\mathbb{R}^3}$ at $\left(0, \frac{\pi}{2}\right)$ get parallel transported to $< 0, 1 > = < 0, 1, 0 >_{\mathbb{R}^3}$ at $\left(\frac{\pi}{2}, 0\right)$.

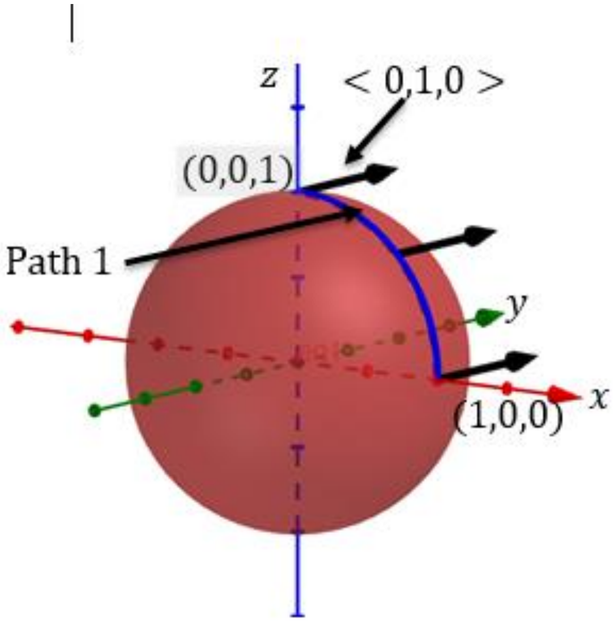
Note: this is different from the vector, $< 1, 0, 0 >_{\mathbb{R}^3}$, from part 1.

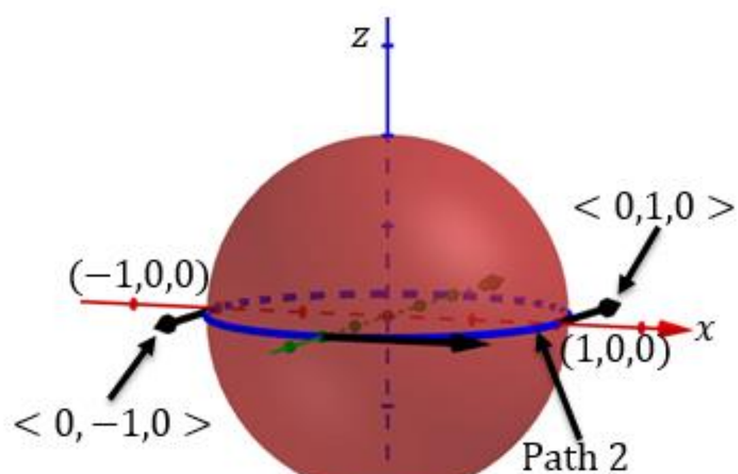
Ex. Let $S^2 \subseteq \mathbb{R}^3$ be the unit sphere where:

$$\vec{\Phi}(\theta, \phi) = (\cos\theta\sin\phi\,, \sin\theta\sin\phi\,, \cos\phi)$$
$$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi.$$

Parallel transport the vector $< 1, 0 >$ at the point $\left(0, \frac{\pi}{2}\right)$ along

1) Path 1: along $\theta = 0,\ \phi = \frac{\pi}{2} - t,\ 0 \le t \le \frac{\pi}{2}$ (to the north pole).
2) Path 2: along $\theta = t,\ \phi = \frac{\pi}{2},\quad 0 \le t \le 2\pi$ (around the equator).





$< 1, 0 >= \overrightarrow{\Phi}_\theta \left(0, \frac{\pi}{2}\right) =< 0,1,0 >_{\mathbb{R}^3}$.

Along path 1 we have: $\gamma_1(t) = \left(0, \frac{\pi}{2} - t\right)$;

So we have: $\gamma_1^1(t) = 0$ $\quad \gamma_1^2(t) = \frac{\pi}{2} - t$

. $\gamma_1^1(t) = 0$ $\quad \dot{\gamma}_1^2(t) = -1$

$\Gamma_{21}^1 = \Gamma_{12}^1 = \cot\phi, \quad \Gamma_{11}^2 = -(sin\phi)(cos\phi)$, other $\Gamma_{ij}^k = 0$.

$k = 1 \quad \dot{v}^1 + \Gamma_{ij}^1(\dot{\gamma}_1^i)v^j = \dot{v}^1 - \Gamma_{21}^1 v^1 - \Gamma_{22}^1 v^2 = 0$

$\Gamma_{21}^1 = \cot\phi, \quad \Gamma_{22}^1 = 0,$ so we get:

$$\dot{v}^1 - (\cot\phi)v^1 = 0.$$

Since $\phi = \frac{\pi}{2} - t$ we have:

$$\dot{v}^1 - (\cot\left(\frac{\pi}{2} - t\right))v^1 = 0$$
$$\dot{v}^1 - (\tan(t))\, v^1 = 0.$$

$$\dot{v}^1 = (\tan(t))\, v^1$$

$$\frac{\dot{v}^1}{v^1} = \frac{\sin(t)}{\cos(t)} .$$

Integrating both sides we get:

$$\ln(v^1) + c_1 = -\ln(cos(t)) + c_2$$
$$\ln(v^1) = -\ln(cos(t)) + c_3$$

$$\Rightarrow \qquad v^1(t) = e^{-\ln(\cos(t)) + c_3}$$
$$v^1(t) = c_4 \sec(t).$$

$V(0) =< v^1(0), v^2(0) > =< 1,0 >$, so we have:

$$1 = v^1(0) = c_4 \sec(0) = c_4 \quad \text{and}$$

$$v^1(t) = \sec(t).$$

$k = 2$ $$\dot{v}^2 + \Gamma^2_{ij}(\dot{\gamma}^i_1)v^j = \dot{v}^2 - \Gamma^2_{21}v^1 - \Gamma^2_{22}v^2 = 0.$$

But $\Gamma^2_{21} = \Gamma^2_{22} = 0$, so we have:

$$\dot{v}^2 = 0 \quad \Rightarrow \quad v^2(t) = c.$$
$$v^2(0) = 0 \quad \Rightarrow \quad v^2(t) = 0.$$

Thus the parallel vector field in the basis $\{\overrightarrow{\Phi}_\theta, \overrightarrow{\Phi}_\phi\}$ is:

$$V(t) = < \sec(t)\,, 0 >.$$

In $\mathbb{R}^3$ this becomes:

$$\begin{aligned} V(t) &= < \sec(t)\,, 0 >= (\sec(t))\overrightarrow{\Phi}_\theta(t) \\ &= (\sec(t))(< -\sin(\theta)(\sin(\phi)),\ \cos(\theta)\,(\sin(\,\phi)), 0 >. \end{aligned}$$

$$\theta = 0, \qquad \phi = \frac{\pi}{2} - t.$$

$$\begin{aligned} V(t) &= (\sec\,(t))(< 0, \cos(0)\left(\sin\left(\frac{\pi}{2} - t\right)\right), 0 >) \\ &= (\sec(t))(< 0, \cos(t)\,, 0 > =< 0,1,0 >_{\mathbb{R}^3}. \end{aligned}$$

So with respect to the standard basis in $\mathbb{R}^3$ we have:

$$V(t) =< 0,1,0 >.$$

Along path 2 we have:

$$\theta = t, \quad \phi = \frac{\pi}{2}; \text{ ie } \gamma_2(t) = \left(t, \frac{\pi}{2}\right) \Rightarrow \gamma_2^1(t) = t, \quad \gamma_2^2(t) = \frac{\pi}{2}$$

$$\dot{\gamma}_2^1 = 1 \qquad \dot{\gamma}_2^2 = 0.$$

$$k = 1 \qquad \dot{v}^1 + \Gamma_{ij}^1(\dot{\gamma}_1^i)v^j = \dot{v}^1 + \Gamma_{11}^1 v^1 + \Gamma_{12}^1 v^2 = 0;$$

$$\Gamma_{11}^1 = 0, \ \Gamma_{12}^1 = \cot\phi, \quad \text{but } \phi = \frac{\pi}{2}, \text{ so } \cot\phi = 0.$$

$$\Rightarrow \qquad \dot{v}^1 = 0 \quad \Rightarrow \quad v^1(t) = c.$$

But $v^1(0) = 1$ so $\quad v^1(t) = 1.$

$$k = 2 \qquad \dot{v}^2 + \Gamma_{ij}^2(\dot{\gamma}_1^i)v^j = \dot{v}^2 - \Gamma_{11}^2 v^1 - \Gamma_{12}^2 v^2 = 0.$$

$$\Gamma_{11}^2 = -(sin\phi)(cos\phi), \qquad \Gamma_{12}^2 = 0.$$

$$\dot{v}^2 - (sin\phi)(cos\phi) = 0.$$

But $\phi = \frac{\pi}{2}$ so $cos\phi = 0.$

$$\dot{v}^2 = 0 \quad \Rightarrow \quad v^2(t) = c.$$

$$v^2(0) = 0 \quad \Rightarrow \quad v^2(t) = 0.$$

Thus we have in the basis $\{\vec{\Phi}_\theta, \vec{\Phi}_\phi\}$:

$$V(t) =< 1{,}0 >.$$

In the standard basis for $\mathbb{R}^3$ we have:

$$\begin{aligned} V(t) &=< 1{,}0 >= \vec{\Phi}_\theta(t) \\ &=< -\sin(\theta)\,(\sin(\phi)),\ \cos(\theta)\,(\sin(\phi)), 0 > \\ &=< -\sin(t)\,, \cos(t)\,, 0 >. \end{aligned}$$