Vector Fields Along Curves- HW Problems

1. Let S^2 be the unit sphere in $\mathbb{R}^3$. On the coordinate patch $(x^1, x^2) \in (0,2\pi) \times (0,\pi)$ the metric is given by

$$g = \begin{pmatrix} \sin^2 x^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

and the Christoffel symbols are:

$\Gamma^1_{21} = \Gamma^1_{12} = cotx^2$, $\Gamma^2_{11} = -sinx^2 cosx^2$, $\Gamma^i_{jk} = 0$ otherwise.

Let p be the point on the sphere given by $x^1 = x^1_0$, $x^2 = x^2_0$.
Let $V(t)$ be a vector field on S^2 with $V(p) =< V^1_0, V^2_0 >$.

Calculate the parallel transport vector field
$V(t) =< V^1(t), V^2(t) >$ along the curve
$\gamma(t) = (x^1_0, t)$, using the initial condition $\gamma(t_0) = (x^1_0, x^2_0)$.

Show that the length of $V(t)$ with respect to g is constant.

2. Let S be the portion of the cone in $\mathbb{R}^3$ given by

$$\vec{\Phi}(u,v) = (ucosv, usinv, u), \quad u > 0.$$

The induced metric is $g = \begin{pmatrix} 2 & 0 \\ 0 & u^2 \end{pmatrix}$ and the non-zero Christoffel symbols are $\Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{u}, \quad \Gamma^1_{22} = -\frac{u}{2}$.

Let $\gamma(t) = (u_0, t)$, be the circle of radius u_0 in local cooradinates (u, v). At $t = 0$ let $V(0) =< V_0^1, V_0^2 >$ be a vector in $T_{\vec{\Phi}(u_0,0)}S$.

Write down the explicit differential equations, but DON'T solve them, that describe the parallel transported vector field $V(t)$.