# Connections and Covariant Differentiation

We saw an example earlier of a vector field being a contravariant vector. This is true in general for vector fields on a manifold. Let $\chi(M)$ be the space of all smooth vector fields on an $n$-dimensional manifold, $M$. If $X \in \chi(M)$, then in local coordinates $x^1, \dots, x^n$ we can write:

$$X = \sum_{i=1}^{n} A^i(x) \frac{\partial}{\partial x^i} \ .$$

Recall that if $\vec{\Phi}: U \subseteq \mathbb{R}^n \to M \subseteq \mathbb{R}^k$, then $\frac{\partial}{\partial x^i}$ means $\frac{\partial \vec{\Phi}}{\partial x^i}$.

If $\bar{x}^1, \dots, \bar{x}^n$ are another set of local coordinates, then by the Chain Rule we can write:

$$\frac{\partial}{\partial x^i} = \sum_{j=1}^{n} \frac{\partial \bar{x}^j}{\partial x^i} \frac{\partial}{\partial \bar{x}^j} = \frac{\partial \bar{x}^j}{\partial x^i} \frac{\partial}{\partial \bar{x}^j}$$

Thus:

$$X = \sum_{i=1}^{n} A^i(x) \frac{\partial}{\partial x^i} = \sum_{i=1}^{n} \sum_{j=1}^{n} A^i \ \frac{\partial \bar{x}^j}{\partial x^i} \frac{\partial}{\partial \bar{x}^j} = \sum_{j=1}^{n} \left( \sum_{i=1}^{n} A^i \frac{\partial \bar{x}^j}{\partial x^i} \right) \frac{\partial}{\partial \bar{x}^j}$$

So we know:

$$\bar{A}^j = \sum_{i=1}^{n} A^i \frac{\partial \bar{x}^j}{\partial x^i}$$

and $X$ is a contravariant vector.

However, what happens if we differentiate a vector field? Is that also a tensor?

We know that a vector field:

$$X = \sum_{i=1}^{n} A^i(x) \frac{\partial}{\partial x^i}$$

is a contravariant vector, so if we change coordinates we get:

$$\bar{A}^j(\bar{x}) = \sum_{i=1}^{n} A^i \frac{\partial \bar{x}^j}{\partial x^i} = A^i \frac{\partial \bar{x}^j}{\partial x^i}$$

Now let's differentiate this equation with respect to $\bar{x}^k$.

$$\frac{\partial \bar{A}^j}{\partial \bar{x}^k} = A^i \frac{\partial}{\partial \bar{x}^k}\left(\frac{\partial \bar{x}^j}{\partial x^i}\right) + \frac{\partial \bar{x}^j}{\partial x^i} \frac{\partial A^i}{\partial \bar{x}^k}$$

Applying the Chain Rule to $\frac{\partial}{\partial \bar{x}^k}\left(\frac{\partial \bar{x}^j}{\partial x^i}\right)$ and $\frac{\partial A^i}{\partial \bar{x}^k}$ we get:

$$\frac{\partial \bar{A}^j}{\partial \bar{x}^k} = A^i \frac{\partial^2 \bar{x}^j}{\partial x^l \partial x^i} \frac{\partial x^l}{\partial \bar{x}^k} + \frac{\partial \bar{x}^j}{\partial x^i} \frac{\partial A^i}{\partial x^l} \frac{\partial x^l}{\partial \bar{x}^k}$$

$$\frac{\partial \bar{A}^j}{\partial \bar{x}^k} = A^i \frac{\partial^2 \bar{x}^j}{\partial x^l \partial x^i} \frac{\partial x^l}{\partial \bar{x}^k} + \frac{\partial \bar{x}^j}{\partial x^i} \frac{\partial x^l}{\partial \bar{x}^k} \frac{\partial A^i}{\partial x^l}.$$

The first term on the right hand side shows us that $\left\{\frac{\partial \bar{A}^j}{\partial \bar{x}^k}\right\}$ is not a tensor unless $\frac{\partial^2 \bar{x}^j}{\partial x^l \partial x^i} = 0$ for all $j, l, i = 1, \dots, n.$

We would like to have a notion of a derivative of a tensor that is also a tensor. This leads us to the idea of connections and covariant derivatives.

Def. Let $M$ be a smooth manifold and let $\chi(M)$ be the space of smooth vector fields on $M$. A **connection on the tangent bundle**, $TM$, is a map:

$$\nabla : \chi(M) \times \chi(M) \to \chi(M)$$

written $\nabla_X Y$, instead of $\nabla(X, Y)$, that satisfies the following:

1) For all $Y \in \chi(M), \nabla(\_\_, Y)$ is linear over $C^\infty(M)$, i.e. for all $f, g \in C^\infty(M)$:

$$\nabla_{(fX_1 + gX_2)} Y = f\left(\nabla_{X_1} Y\right) + g\left(\nabla_{X_2} Y\right)$$

2) For all vector fields $X \in \chi(M), \nabla(X, \_\_)$ is linear over $\mathbb{R}$, i.e. for all $a, b \in \mathbb{R}$:

$$\nabla_X (aY_1 + bY_2) = a(\nabla_X Y_1) + b(\nabla_X Y_2)$$

3) For all vector fields $X \in \chi(M), \nabla(X, \_\_)$ satisfies the Product Rule:

$$\nabla_X (fY) = (Xf)Y + f(\nabla_X Y)$$

for all $f \in C^\infty(M)$.

The vector field $\nabla_X Y$ is called the **covariant derivative** of $Y$ in the direction of $X$.

Earlier we defined the directional derivative of a real valued function, $f$, in the direction of a vector. The covariant derivative of a vector field, $Y$, in the direction of $X$ at a point $p \in M, \nabla_X Y|_p$, is really the directional derivative of $Y$ at point $p$ in the direction $X_p$.

Thus, a covariant derivative of a vector field is a generalization of the directional derivative of a real valued function. We will say $\nabla_X f = X(f)$, i.e. the covariant derivative of a function $f: M \to \mathbb{R}$ in the direction of $X \in \chi(M)$ is the directional derivative of $f$ in the direction of $X$.

Over a coordinate patch $U \subseteq M, \nabla$ is completely determined once we know the values for $X = \partial_i = \frac{\partial}{\partial x^i}$ and $Y = \partial_j = \frac{\partial}{\partial x^j}$ . Since $\nabla_{\partial_i}(\partial_j)$ is a vector field on $M$, we can write:

$$\nabla_{\partial_i}(\partial_j) = \Gamma_{ij}^k \partial_k = \sum_{k=1}^{n} \Gamma_{ij}^k \partial_k .$$

The components of this vector field, $\Gamma_{ij}^k$, are smooth real valued functions on $M$. $\nabla_{\partial_i}(\partial_j)$ is the projection of $\frac{\partial}{\partial x^i}(\frac{\partial \vec{\Phi}}{\partial x^j})$ on to the tangent space spanned by $\left\{\frac{\partial \vec{\Phi}}{\partial x^1}, \dots, \frac{\partial \vec{\Phi}}{\partial x^n}\right\}$.

Def. The functions $\Gamma_{ij}^k$ are called the **Christoffel symbols** (or the Christoffel symbols of the second kind) of the connection $\nabla$.

By using the properties of a connection $\nabla$ in its definition we can now find a formula for $\nabla_X Y$ on a coordinate patch, $U \subseteq M$, in terms of the components of $X, Y$ and the Christoffel symbols $\Gamma_{ij}^k$.

Ex. Let $x^1, \dots, x^n$ be local coordinates on $U \subseteq M$ and:

$$X = \sum_{i=1}^{n} X^i \, \partial_i \qquad Y = \sum_{j=1}^{n} Y^j \, \partial_j$$

Find $\nabla_X Y$.

$$\begin{aligned}
\nabla_X Y &= \nabla_{X^i \partial_i}\left(Y^j \partial_j\right) \\
&= \left[X^i \partial_i\left(Y^j\right)\right]\partial_j + Y^j \nabla_{X^i \partial_i}\left(\partial_j\right) && \text{by Property \#3} \\
&= \left[X^i \partial_i\left(Y^j\right)\right]\partial_j + Y^j X^i \nabla_{\partial_i}\left(\partial_j\right) && \text{by Property \#1} \\
&= \left[X^i \partial_i\left(Y^j\right)\right]\partial_j + Y^j X^i \left(\Gamma_{ij}^k \partial_k\right)
\end{aligned}$$

Now switch the roll of $j$ and $k$ in the second term:

$$= \left[X^i \partial_i\left(Y^j\right)\right]\partial_j + Y^k X^i \left(\Gamma_{ik}^j \partial_j\right)$$

$$\nabla_X Y = \left(X^i \partial_i\left(Y^j\right) + Y^k X^i \left(\Gamma_{ik}^j\right)\right) \partial_j .$$

In particular if $X = \partial_t = \dfrac{\partial}{\partial x^t}$ , then we have:

$$\nabla_{\partial_t} Y = \left(\frac{\partial}{\partial x^t}\left(Y^j\right) + \Gamma_{tk}^j Y^k\right) \partial_j .$$

Def. We call the $(1,1)$ tensor with components:

$$\nabla Y = \left(\frac{\partial}{\partial x^t}\left(Y^j\right) + \Gamma^j_{tk} Y^k\right)$$

the **covariant derivative** of the $(1,0)$ tensor $Y$.

So if we have a local coordinate system $U \subseteq M, (x^1, \ldots, x^n)$, when taking the covariant derivative of a vector field $Y$, we differentiate each component $Y^j$ with respect to $x^t$ but then add a second term, $\Gamma^j_{tk} Y^k$, where:

$$\Gamma^j_{tk} \partial_j = \nabla_{\partial_t}(\partial_k).$$

Where did this second term come from? In other words, why isn't the derivative of $Y$ equal to $\left(\frac{\partial Y^j}{\partial x^t}\right)$ as it was in second year calculus?

The "problem" we have with a vector field on a general smooth manifold is that the basis of the tangent space is also a function of $(x^1, \ldots, x^n)$ (the basis of $T_p(\mathbb{R}^n)$ is $\vec{e}_1, \ldots, \vec{e}_n$ for <u>every</u> point $p$). The covariant derivative of a vector field is measuring the rate of change of the components of the vector field, $\left(\frac{\partial Y^j}{\partial x^t}\right)$, as well as the rate of change of the basis of the tangent space, $\Gamma^j_{tk} Y^k$.

Ex. Suppose the surface $S \subseteq \mathbb{R}^3$ is given by:

$$\vec{\Phi}(x^1, x^2) = (x^1, x^2, (x^1)^2 + (x^2)^2).$$

Then a basis for the tangent space at $\vec{\Phi}(x^1, x^2)$ is given by:

$$\vec{\Phi}_{x^1} = (1, 0, 2x^1) \qquad \vec{\Phi}_{x^2} = (0, 1, 2x^2).$$

So the basis changes as $\vec{\Phi}(x^1, x^2)$ changes.

In general, if we have a vector field $Y = Y^i \partial_i$, where $\partial_i = \dfrac{\partial \vec{\Phi}}{\partial x^i}$, then

$$\frac{\partial Y}{\partial x^k} = \frac{\partial}{\partial x^k}\left(Y^i \partial_i\right) = \frac{\partial Y^i}{\partial x^k}\partial_i + Y^i \frac{\partial}{\partial x^k}(\partial_i)$$
$$= \frac{\partial Y^i}{\partial x^k}\partial_i + Y^i \Gamma_{ik}^j \partial_j \,.$$

Now reindex the second term by interchanging $i$ and $j$.

$$\frac{\partial Y}{\partial x^k} = \frac{\partial Y^i}{\partial x^k}\partial_i + Y^j \Gamma_{jk}^i \partial_i$$

$$\frac{\partial Y}{\partial x^k} = \left(\frac{\partial Y^i}{\partial x^k} + Y^j \Gamma_{jk}^i\right)\partial_i \,.$$

Ex. If we take a vector field on $\mathbb{R}^n$ where:

$$\vec{f}(x^1, \dots, x^n) = < f^1(x^1, \dots, x^n), \dots, f^n(x^1, \dots, x^n) >$$

and $\partial_i = \vec{e}_i = < 0, \dots, 1, \dots, 0 >$, the standard basis on $\mathbb{R}^n$, then:

$$\vec{f}(x^1, \dots, x^n) = f^j \partial_j \text{ and } \nabla_{\partial_i} \vec{f} = \frac{\partial f^j}{\partial x^i}\partial_j \,.$$

In this case, $\nabla_{\partial_i} \vec{f}$, is just the usual partial derivative of a vector field on $\mathbb{R}^n$ (i.e. the derivative of a vector field that one learns in second year calculus). Here, all of the Christoffel symbols are equal to $0$.

Notice: In the example above, the covariant derivative $\nabla \vec{f}$ is just the Jacobian of $\vec{f}$. That is, the components of $\nabla \vec{f}$ are $\left(\dfrac{\partial f^i}{\partial x^j}\right)$.

So far we have a definition for the covariant derivative of a $(1,0)$ tensor, with components $Y^i$, whose components we can write as:

$$Y^i_{;k} = \frac{\partial Y^i}{\partial x^k} + \Gamma^i_{kj} Y^j.$$

In general, if $F$ is an $(r,s)$ tensor over a manifold, $M$, and has components $F^{i_1 \dots i_r}_{j_1 \dots j_s}$ over a coordinate chart $U$, then the components of the covariant derivative of $F$, $\nabla F$, are:

$$F^{i_1 \dots i_r}_{j_1 \dots j_s;k} = \frac{\partial F^{i_1 \dots i_r}_{j_1 \dots j_s}}{\partial x^k} + \sum_{\alpha=1}^{r} \Gamma^{i_\alpha}_{kp} F^{i_1 \dots i_{\alpha-1} p i_{\alpha+1} \dots i_r}_{j_1 \dots j_s} - \sum_{\beta=1}^{s} \Gamma^{p}_{k j_\beta} F^{i_1 \dots \dots i_r}_{j_1 \dots j_{\beta-1} p j_{\beta+1} \dots j_s}$$

$\nabla F$ is an $(r, s+1)$ tensor.

Ex. If $F$ is a $(2,1)$ tensor with components $F^{ij}_k$, then $\nabla F$ is a $(2,2)$ tensor with components:

$$F^{ij}_{k\,;\,l} = \frac{\partial F^{ij}_k}{\partial x^l} + \Gamma^i_{lp} F^{pj}_k + \Gamma^j_{lp} F^{ip}_k - \Gamma^p_{lk} F^{ij}_p \ .$$

We now state a few results whose proofs can be found in the appendix to Connections and Covariant Differentiation.

$\Gamma_{ij}^{k}$ is not a tensor, but how do the components change under a change of coordinates?

Proposition:

Let $U$ and $\overline{U}$ be overlapping coordinate patches on a manifold, $M$, with local coordinates $(x^1, \dots, x^n)$ and $(\bar{x}^1, \dots, \bar{x}^n)$ respectively, then

$$\bar{\Gamma}_{ij}^{k} = \frac{\partial x^r}{\partial \bar{x}^j} \frac{\partial x^l}{\partial \bar{x}^i} \frac{\partial \bar{x}^k}{\partial x^m} \Gamma_{lr}^{m} + \frac{\partial^2 x^m}{\partial \bar{x}^i \, \partial \bar{x}^j} \frac{\partial \bar{x}^k}{\partial x^m}$$

Proof (see appendix).

Proposition: $T_{i\,;k} = \frac{\partial T_i}{\partial x^k} - \Gamma_{ik}^{j} T_j$ are the components of a $(0,2)$ tensor.

Proof (see appendix).

Levi-Civita Theorem:

Let $(M, g)$ be a Riemannian manifold. There exists a unique connection $\nabla$ that satisfies the following:

1) $\nabla g = 0$
*2)* For all $X, Y \in \chi(M)$, $[X, Y] = \nabla_X Y - \nabla_Y X$.

Proof (see appendix).

Proposition: Let $(M, g)$ be a smooth manifold. Then, over a coordinate patch $U \subseteq M$ with local coordinates $(x^1, \dots, x^n)$ the Christoffel symbols of the Levi-Civita connection are given by

$$\Gamma^i_{jk} = \sum_{l=1}^{n} \frac{1}{2} g^{il} \left( \frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right)$$

where $g^{ij}$ are the entries to the inverse matrix $(g_{kl})$.

Proof (see appendix).

Ex. Using the formula for $\Gamma^i_{jk}$ in terms of the metric $g$, show: $\Gamma^i_{jk} = \Gamma^i_{kj}$ .

$$\Gamma^i_{jk} = \sum_{l=1}^{n} \frac{1}{2} g^{il} \left( \frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right)$$

$$\Gamma^i_{kj} = \sum_{l=1}^{n} \frac{1}{2} g^{il} \left( \frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{kj}}{\partial x^l} \right)$$

These two expressions are equal because $g_{\alpha\beta} = g_{\beta\alpha}$.

Ex. Using the formula for $\Gamma^i_{jk}$ in terms of the metric $g$, show that:

$$\nabla_k g_{ij} = g_{ij;k} = 0.$$

$$g_{ij;k} = \frac{\partial g_{ij}}{\partial x^k} - \Gamma^p_{ik} g_{pj} - \Gamma^p_{jk} g_{ip}$$

$$= \frac{\partial g_{ij}}{\partial x^k} - \frac{1}{2}\sum_{l=1}^{n} g_{pj}\, g^{pl}\left(\frac{\partial g_{kl}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^l}\right)$$
$$- \frac{1}{2}\sum_{l=1}^{n} g_{ip}\, g^{pl}\left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right)$$

$$= \frac{\partial g_{ij}}{\partial x^k} - \frac{1}{2}\sum_{l=1}^{n} \delta^l_j\left(\frac{\partial g_{kl}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^l}\right) - \frac{1}{2}\sum_{l=1}^{n} \delta^l_i\left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right)$$

$$= \frac{\partial g_{ij}}{\partial x^k} - \frac{1}{2}\left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ji}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^j}\right) - \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i}\right).$$

Since $g_{\alpha\beta} = g_{\beta\alpha}$, we have:

$$g_{ij;k} = \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ij}}{\partial x^k} = 0.$$

Ex. Let $g$ be the metric on $\mathbb{R}^2$ induced by $\vec{\Phi}(x^1, x^2) = (x^1 cos x^2,\ x^1 sin x^2)$. Find the eight Christoffel symbols directly from the metric $g$. Given the vector field $V = x^2\vec{\Phi}_{x^1} - x^1\vec{\Phi}_{x^2} =< x^2, -x^1 >$, find the components of $\nabla V$.

$\vec{\Phi}_{x^1} =< cos x^2, sin x^2 >$ , $\quad \vec{\Phi}_{x^2} =< -x^1 sin x^2,\ x^1 cos x^2 >$

$g_{11} = \vec{\Phi}_{x^1} \cdot \vec{\Phi}_{x^1} = 1$ ; $\quad g_{12} = g_{21} = \vec{\Phi}_{x^1} \cdot \vec{\Phi}_{x^2} = 0$ ;

$g_{22} = \vec{\Phi}_{x^2} \cdot \vec{\Phi}_{x^2} = (x^1)^2$

So we have: $\quad (g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & (x^1)^2 \end{pmatrix}$ and $\quad (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{(x^1)^2} \end{pmatrix}$.

$$\Gamma_{jk}^i = \sum_{l=1}^n \frac{1}{2} g^{il}\left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right)$$

But the only non-zero derivative of $g_{ij}$ is $\frac{\partial g_{22}}{\partial x^1} = 2x^1$.

The only $\Gamma_{jk}^i$ that include $\frac{\partial g_{22}}{\partial x^1}$ are $\Gamma_{21}^1 = \Gamma_{12}^1,\ \Gamma_{22}^1,\ \Gamma_{21}^2 = \Gamma_{12}^2$ and $\Gamma_{22}^2$.

$$\Gamma_{21}^1 = \Gamma_{12}^1 = \sum_{l=1}^2 \frac{1}{2} g^{1l}\left(\frac{\partial g_{2l}}{\partial x^1} + \frac{\partial g_{l1}}{\partial x^2} - \frac{\partial g_{12}}{\partial x^l}\right) = 0; \quad \text{since } g^{12} = 0$$

$$\Gamma_{22}^1 = \sum_{l=1}^2 \frac{1}{2} g^{1l}\left(\frac{\partial g_{2l}}{\partial x^2} + \frac{\partial g_{l2}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^l}\right) = \frac{1}{2}(g^{11})\left(-\frac{\partial g_{22}}{\partial x^1}\right) = -x^1$$

$$\Gamma_{21}^2 = \Gamma_{12}^2 = \sum_{l=1}^2 \frac{1}{2} g^{2l}\left(\frac{\partial g_{2l}}{\partial x^1} + \frac{\partial g_{l1}}{\partial x^2} - \frac{\partial g_{12}}{\partial x^l}\right) = \frac{1}{2}(g^{22})\left(\frac{\partial g_{22}}{\partial x^1}\right) = \frac{1}{x^1}$$

$$\Gamma_{22}^2 = \sum_{l=1}^2 \frac{1}{2} g^{2l}\left(\frac{\partial g_{2l}}{\partial x^2} + \frac{\partial g_{l2}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^l}\right) = 0; \quad \text{since } g^{21} = 0.$$

All other $\Gamma_{jk}^i = 0$.

$\nabla_k V^i = \frac{\partial V^i}{\partial x^k} + \Gamma^i_{jk} V^j$ ; where $V =< x^2, -x^1 >=< V^1, V^2 >$.

If we call $T = \nabla V$ then we have:

$$T^1_1 = \nabla_1 V^1 = \frac{\partial V^1}{\partial x^1} + \Gamma^1_{j1} V^j = 0$$

$$T^1_2 = \nabla_2 V^1 = \frac{\partial V^1}{\partial x^2} + \Gamma^1_{j2} V^j = 1 + \Gamma^1_{22} V^2 = 1 + (x^1)^2$$

$$T^2_1 = \nabla_1 V^2 = \frac{\partial V^2}{\partial x^1} + \Gamma^2_{j1} V^j = -1 + \Gamma^2_{21} V^2 = -1 + \left(\frac{1}{x^1}\right)(-x^1) = -2.$$

$$T^2_2 = \nabla_2 V^2 = \frac{\partial V^2}{\partial x^2} + \Gamma^2_{j2} V^j = \Gamma^2_{12} V^1 = (\tfrac{1}{x^1})(x^2) .$$

Using the fact that the covariant derivative of a contravariant vector is just the projection of the usual derivative onto the tangent space we have another way to calculate the Christoffel symbols.

Ex. Let $S^2 \subseteq \mathbb{R}^3$ be the unit sphere where:

$$\overrightarrow{\Phi}(x^1, x^2) = (\cos x^1 \sin x^2 , \sin x^1 \sin x^2 , \cos x^2)$$
$$0 \leq x^1 \leq 2\pi , \quad 0 \leq x^2 \leq \pi.$$

We know that $\nabla_{\partial_i}(\partial_j) = \sum_{k=1}^{n} \Gamma^k_{ij}\, \partial_k$.

Use the fact that the coefficient of the projection of $\overrightarrow{\Phi}_{x^i x^j}$ on to $\overrightarrow{\Phi}_{x^k} = \partial_k$ is $\Gamma^k_{ij}$ to show that the Christoffel symbols are
$\Gamma^1_{21} = \Gamma^1_{12} = cot x^2$, $\Gamma^2_{11} = -(sin x^2)(cos x^2)$, $\Gamma^k_{ij} = 0$ otherwise.
(For HW you need to do this calculation with the formula for $\Gamma^k_{ij}$ in terms of the metric).

$\partial_1 = \overrightarrow{\Phi}_{x^1} =< -(sinx^1)sinx^2, (cosx^1)sinx^2, 0 >;$
$\partial_2 = \overrightarrow{\Phi}_{x^2} =< (cosx^1)cosx^2, (sinx^1)cosx^2, -sinx^2 >$

$\overrightarrow{\Phi}_{x^1} \cdot \overrightarrow{\Phi}_{x^1} = \sin^2 x^2$
$\overrightarrow{\Phi}_{x^2} \cdot \overrightarrow{\Phi}_{x^2} = 1$

$\overrightarrow{\Phi}_{x^1x^1} =< -(cosx^1)sinx^2, -(sinx^1)sinx^2, 0 >$
$\overrightarrow{\Phi}_{x^1x^2} =< -(sinx^1)cosx^2, (cosx^1)cosx^2, 0 >$
$\overrightarrow{\Phi}_{x^2x^2} =< -(cosx^1)sinx^2, -(sinx^1)sinx^2, -cosx^2 >$

Recall from 2nd year calculus that the projection of a vector $\vec{w}$ onto a vector $\vec{v}$ is given by: $Proj_{\vec{v}}\vec{w} = (\frac{\vec{v}\cdot\vec{w}}{\vec{v}\cdot\vec{v}})\vec{v}.$

$$\nabla_{\partial_1}(\partial_1) = \sum_{k=1}^{n} \Gamma_{11}^{k}\, \partial_k = \Gamma_{11}^{1}\partial_1 + \Gamma_{11}^{2}\partial_2$$

$$\Gamma_{11}^{1} = \frac{\overrightarrow{\Phi}_{x^1x^1}\cdot\overrightarrow{\Phi}_{x^1}}{\overrightarrow{\Phi}_{x^1}\cdot\overrightarrow{\Phi}_{x^1}}$$
$$= \frac{< -(cosx^1)sinx^2, -(sinx^1)sinx^2, 0 > \cdot < -(sinx^1)sinx^2, (cosx^1)sinx^2, 0 >}{\sin^2 x^2}$$
$$= 0$$

$$\Gamma_{11}^{2} = \frac{\overrightarrow{\Phi}_{x^1x^1}\cdot\overrightarrow{\Phi}_{x^2}}{\overrightarrow{\Phi}_{x^2}\cdot\overrightarrow{\Phi}_{x^2}}$$
$$= \frac{< -(cosx^1)sinx^2, -(sinx^1)sinx^2, 0 > \cdot < (cosx^1)cosx^2, (sinx^1)cosx^2, -sinx^2 >}{1}$$
$$= -(sinx^2)cosx^2$$

$$\nabla_{\partial_2}(\partial_1) = \sum_{k=1}^{n} \Gamma_{21}^{k}\, \partial_k = \Gamma_{21}^{1}\partial_1 + \Gamma_{21}^{2}\partial_2$$

$$\Gamma_{21}^{1} = \Gamma_{12}^{1} = \frac{\vec{\Phi}_{x^1x^2}\cdot\vec{\Phi}_{x^1}}{\vec{\Phi}_{x^1}\cdot\vec{\Phi}_{x^1}}$$

$$= \frac{<-(sinx^1)cosx^2,(cosx^1)cosx^2,0>\cdot<-(sinx^1)sinx^2,(cosx^1)sinx^2,0>}{\sin^2 x^2}$$

$$= \frac{(cosx^2)sinx^2}{\sin^2 x^2} = cotx^2$$

$$\Gamma_{21}^{2} = \Gamma_{12}^{2} = \frac{\vec{\Phi}_{x^1x^2}\cdot\vec{\Phi}_{x^2}}{\vec{\Phi}_{x^2}\cdot\vec{\Phi}_{x^2}}$$

$$= \frac{<-(sinx^1)cosx^2,(cosx^1)cosx^2,0>\cdot<(cosx^1)cosx^2,(sinx^1)cosx^2,-sinx^2>}{1}$$

$$= 0$$

$$\nabla_{\partial_2}(\partial_2) = \sum_{k=1}^{n} \Gamma_{22}^{k}\, \partial_k = \Gamma_{22}^{1}\partial_1 + \Gamma_{22}^{2}\partial_2$$

$$\Gamma_{22}^{1} = \frac{\vec{\Phi}_{x^2x^2}\cdot\vec{\Phi}_{x^1}}{\vec{\Phi}_{x^1}\cdot\vec{\Phi}_{x^1}}$$

$$= \frac{<-(cosx^1)sinx^2,-(sinx^1)sinx^2,-cosx^2>\cdot<-(sinx^1)sinx^2,(cosx^1)sinx^2,0>}{\sin^2 x^2}$$

$$= 0.$$

$$\Gamma_{22}^{2} = \frac{\vec{\Phi}_{x^2x^2}\cdot\vec{\Phi}_{x^2}}{\vec{\Phi}_{x^2}\cdot\vec{\Phi}_{x^2}}$$

$$= \frac{<-(cosx^1)sinx^2,-(sinx^1)sinx^2,-cosx^2>\cdot<(cosx^1)cosx^2,(sinx^1)cosx^2,-sinx^2>}{1}$$

$$= 0.$$