Appendix to Connections and Covariant Differentiation

Γ_{ij}^k is not a tensor, but how do the components change under a change of coordinates?

Proposition:

Let U and $\overline{U}$ be overlapping coordinate patches on a manifold, M, with local coordinates $(x^1, \dots, x^n)$ and $(\bar{x}^1, \dots, \bar{x}^n)$ respectively, then

$$\bar{\Gamma}_{ij}^k = \frac{\partial x^r}{\partial \bar{x}^j} \frac{\partial x^l}{\partial \bar{x}^i} \frac{\partial \bar{x}^k}{\partial x^m} \Gamma_{lr}^m + \frac{\partial^2 x^m}{\partial \bar{x}^i \partial \bar{x}^j} \frac{\partial \bar{x}^k}{\partial x^m}$$

Proof: $\nabla_{\frac{\partial}{\partial \bar{x}^i}} \left(\frac{\partial}{\partial \bar{x}^j} \right) = \bar{\Gamma}_{ij}^k \, \bar{\partial}_k.$

By the Chain Rule:

$$\frac{\partial}{\partial \bar{x}^i} = \sum_{l=1}^n \frac{\partial x^l}{\partial \bar{x}^i} \frac{\partial}{\partial x^l}; \qquad \frac{\partial}{\partial \bar{x}^j} = \sum_{m=1}^n \frac{\partial x^m}{\partial \bar{x}^j} \frac{\partial}{\partial x^m}$$

so $\nabla_{\frac{\partial}{\partial \bar{x}^i}} \left(\frac{\partial}{\partial \bar{x}^j} \right) = \nabla_{\frac{\partial x^l}{\partial \bar{x}^i} \frac{\partial}{\partial x^l}} \left(\frac{\partial x^m}{\partial \bar{x}^j} \frac{\partial}{\partial x^m} \right).$

By Property #3 in the definition of a connection we get:

$$\bar{\Gamma}_{ij}^{k}\frac{\partial}{\partial \bar{x}^{k}} = \frac{\partial x^{l}}{\partial \bar{x}^{i}}\frac{\partial}{\partial x^{l}}\left(\frac{\partial x^{m}}{\partial \bar{x}^{j}}\right)\frac{\partial}{\partial x^{m}} + \frac{\partial x^{m}}{\partial \bar{x}^{j}}\nabla_{\frac{\partial x^{l}}{\partial \bar{x}^{i}}\frac{\partial}{\partial x^{l}}}\left(\frac{\partial}{\partial x^{m}}\right).$$

By Property #1 we get:

$$= \frac{\partial x^{l}}{\partial \bar{x}^{i}}\frac{\partial^{2} x^{m}}{\partial x^{l}\partial \bar{x}^{j}}\frac{\partial}{\partial x^{m}} + \frac{\partial x^{m}}{\partial \bar{x}^{j}}\frac{\partial x^{l}}{\partial \bar{x}^{i}}\nabla_{\frac{\partial}{\partial x^{l}}}\left(\frac{\partial}{\partial x^{m}}\right)$$

$$= \frac{\partial x^{l}}{\partial \bar{x}^{i}}\frac{\partial^{2} x^{m}}{\partial x^{l}\partial \bar{x}^{j}}\frac{\partial}{\partial x^{m}} + \frac{\partial x^{m}}{\partial \bar{x}^{j}}\frac{\partial x^{l}}{\partial \bar{x}^{i}}\Gamma_{lm}^{t}\frac{\partial}{\partial x^{t}}.$$

Reindex the second term by replacing m with r and t with m:

$$= \frac{\partial x^{l}}{\partial \bar{x}^{i}}\frac{\partial^{2} x^{m}}{\partial x^{l}\partial \bar{x}^{j}}\frac{\partial}{\partial x^{m}} + \frac{\partial x^{r}}{\partial \bar{x}^{j}}\frac{\partial x^{l}}{\partial \bar{x}^{i}}\Gamma_{lr}^{m}\frac{\partial}{\partial x^{m}}$$

Thus $$\bar{\Gamma}_{ij}^{k}\frac{\partial}{\partial \bar{x}^{k}} = \left(\frac{\partial x^{l}}{\partial \bar{x}^{i}}\frac{\partial^{2} x^{m}}{\partial x^{l}\,\partial \bar{x}^{j}} + \frac{\partial x^{r}}{\partial \bar{x}^{j}}\frac{\partial x^{l}}{\partial \bar{x}^{i}}\Gamma_{lr}^{m}\right)\frac{\partial}{\partial x^{m}}.$$

Now apply both sides to $\bar{x}^{k}$:

$$\bar{\Gamma}_{ij}^{k}\frac{\partial \bar{x}^{k}}{\partial \bar{x}^{k}} = \frac{\partial x^{l}}{\partial \bar{x}^{i}}\frac{\partial^{2} x^{m}}{\partial x^{l}\,\partial \bar{x}^{j}}\frac{\partial \bar{x}^{k}}{\partial x^{m}} + \frac{\partial x^{r}}{\partial \bar{x}^{j}}\frac{\partial x^{l}}{\partial \bar{x}^{i}}\frac{\partial \bar{x}^{k}}{\partial x^{m}}\Gamma_{lr}^{m}\,.$$

Notice by the Chain Rule:

$$\frac{\partial^2 x^m}{\partial \bar{x}^i\, \partial \bar{x}^j} = \frac{\partial}{\partial \bar{x}^i}\left(\frac{\partial x^m}{\partial \bar{x}^j}\right) = \frac{\partial^2 x^m}{\partial x^l\, \partial \bar{x}^j}\,\frac{\partial x^l}{\partial \bar{x}^i} \Longrightarrow$$

$$\bar{\Gamma}_{ij}^k = \frac{\partial x^r}{\partial \bar{x}^j}\,\frac{\partial x^l}{\partial \bar{x}^i}\,\frac{\partial \bar{x}^k}{\partial x^m}\,\Gamma_{lr}^m + \frac{\partial^2 x^m}{\partial \bar{x}^i\, \partial \bar{x}^j}\,\frac{\partial \bar{x}^k}{\partial x^m}\,.$$

Now let's use this relationship to show:

Proposition: $$\frac{\partial^2 x^m}{\partial \bar{x}^i\, \partial \bar{x}^j} = \bar{\Gamma}_{ij}^s\,\frac{\partial x^m}{\partial \bar{x}^s} - \frac{\partial x^r}{\partial \bar{x}^j}\frac{\partial x^l}{\partial \bar{x}^i}\Gamma_{lr}^m\ .$$

Proof: By the Chain Rule:

$$\delta_s^k = \frac{\partial \bar{x}^k}{\partial \bar{x}^s} = \frac{\partial \bar{x}^k}{\partial x^m}\frac{\partial x^m}{\partial \bar{x}^s}\ .$$

Now take our previous result:

$$\bar{\Gamma}_{ij}^k = \frac{\partial x^r}{\partial \bar{x}^j}\,\frac{\partial x^l}{\partial \bar{x}^i}\,\frac{\partial \bar{x}^k}{\partial x^m}\,\Gamma_{lr}^m + \frac{\partial^2 x^m}{\partial \bar{x}^i\, \partial \bar{x}^j}\,\frac{\partial \bar{x}^k}{\partial x^m}$$

$$\bar{\Gamma}_{ij}^k - \frac{\partial x^r}{\partial \bar{x}^j}\,\frac{\partial x^l}{\partial \bar{x}^i}\,\frac{\partial \bar{x}^k}{\partial x^m}\,\Gamma_{lr}^m = \frac{\partial^2 x^m}{\partial \bar{x}^i\, \partial \bar{x}^j}\,\frac{\partial \bar{x}^k}{\partial x^m}\,.$$

Now multiply by $\frac{\partial x^m}{\partial \bar{x}^s}$:

$$\bar{\Gamma}_{ij}^k \frac{\partial x^m}{\partial \bar{x}^s} - \frac{\partial x^r}{\partial \bar{x}^j} \frac{\partial x^l}{\partial \bar{x}^i} \frac{\partial \bar{x}^k}{\partial x^m} \frac{\partial x^m}{\partial \bar{x}^s} \Gamma_{lr}^m = \frac{\partial^2 x^m}{\partial \bar{x}^i \partial \bar{x}^j} \frac{\partial \bar{x}^k}{\partial x^m} \frac{\partial x^m}{\partial \bar{x}^s}$$

$$\bar{\Gamma}_{ij}^k \frac{\partial x^m}{\partial \bar{x}^s} - \frac{\partial x^r}{\partial \bar{x}^j} \frac{\partial x^l}{\partial \bar{x}^i} \delta_s^k \Gamma_{lr}^m = \frac{\partial^2 x^m}{\partial \bar{x}^i \partial \bar{x}^j} \delta_s^k$$

$$\bar{\Gamma}_{ij}^s \frac{\partial x^m}{\partial \bar{x}^s} - \frac{\partial x^r}{\partial \bar{x}^j} \frac{\partial x^l}{\partial \bar{x}^i} \Gamma_{lr}^m = \frac{\partial^2 x^m}{\partial \bar{x}^i \partial \bar{x}^j} .$$

Now we can show that:

Proposition: $T_{i\,;k} = \frac{\partial T_i}{\partial x^k} - \Gamma_{ik}^j T_j$ are the components of a $(0, 2)$ tensor.

Proof: T_i are components of a $(0,1)$ tensor, so: $\bar{T}_i = T_j \frac{\partial x^j}{\partial \bar{x}^i}$.

Differentiating both sides we get:

$$\frac{\partial \bar{T}_i}{\partial \bar{x}^k} = \frac{\partial T_j}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^i} + T_j \frac{\partial^2 x^j}{\partial \bar{x}^i \partial \bar{x}^k} .$$

Now by the Chain Rule:

$$= \frac{\partial T_j}{\partial x^r} \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^i} + T_j \left(\frac{\partial^2 x^j}{\partial \bar{x}^i \, \partial \bar{x}^k} \right) .$$

Using $\frac{\partial^2 x^j}{\partial \bar{x}^i \, \partial \bar{x}^k} = \bar{\Gamma}^s_{ik} \frac{\partial x^j}{\partial \bar{x}^s} - \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^l}{\partial \bar{x}^i} \Gamma^j_{lr}$ we get:

$$\frac{\partial \bar{T}_i}{\partial \bar{x}^k} = \frac{\partial T_j}{\partial x^r} \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^i} + T_j \left(\bar{\Gamma}^s_{ik} \frac{\partial x^j}{\partial \bar{x}^s} - \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^l}{\partial \bar{x}^i} \Gamma^j_{lr} \right)$$

$$= \frac{\partial T_j}{\partial x^r} \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^i} + \bar{\Gamma}^s_{ik} T_j \frac{\partial x^j}{\partial \bar{x}^s} - \Gamma^j_{lr} T_j \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^l}{\partial \bar{x}^i}$$

$$\frac{\partial \bar{T}_i}{\partial \bar{x}^k} = \frac{\partial T_j}{\partial x^r} \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^i} + \bar{\Gamma}^s_{ik} \bar{T}_s - \Gamma^j_{lr} T_j \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^l}{\partial \bar{x}^i} .$$

Now if we subtract $\bar{\Gamma}^s_{ik} \bar{T}_s$ from both sides and reindex the last term by switching l and j, we get:

$$\frac{\partial \bar{T}_i}{\partial \bar{x}^k} - \bar{\Gamma}^s_{ik} \bar{T}_s = \left(\frac{\partial T_j}{\partial x^r} - \Gamma^l_{jr} T_l \right) \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^i} .$$

So $\nabla_k T_i = T_{i\,;k} = \frac{\partial T_i}{\partial x^k} - \Gamma^j_{ik} T_j$ are the components of a $(0, 2)$ tensor.

In Riemannian geometry there is a very special connection called the Levi-Civita connection.

Levi-Civita Theorem:

Let (M, g) be a Riemannian manifold. There exists a unique connection ∇ that satisfies the following:

1) $\nabla g = 0$
2) For all $X, Y \in \chi(M)$, $[X, Y] = \nabla_X Y - \nabla_Y X$.

Condition #1, $\nabla g = 0$, is equivalent to saying that the following product rule holds: $\nabla_X(< Y, Z >) = < \nabla_X Y, Z > + < Y, \nabla_X Z >$ where $< X, Y > = g(X, Y)$.

The second condition implies that: $\Gamma^i_{jk} = \Gamma^i_{kj}$.

Proof: Let $\nabla_Z(< X, Y >) = Z(X, Y)$. If the connection exists, then $\nabla g = 0$ and we have through the product rule:

$$* \qquad X(Y, Z) = < \nabla_X Y, Z > + < Y, \nabla_X Z >$$

$$** \qquad Y(Z, X) = < \nabla_Y Z, X > + < Z, \nabla_Y X >$$

$$*** \qquad Z(X, Y) = < \nabla_Z X, Y > + < X, \nabla_Z Y >$$

Now if we add $*$ and $**$ while subtracting $***$, we get:

$$X(Y,Z) + Y(Z,X) - Z(X,Y)$$

$$= < \nabla_X Y - \nabla_Y X, Z > + < \nabla_X Z - \nabla_Z X, Y >$$
$$+ < \nabla_Y Z - \nabla_Z Y, X > +2 < Z, \nabla_Y X >.$$

Using the fact that ∇ is symmetric, meaning $[X,Y] = \nabla_X Y - \nabla_Y X$, we can write:

$$X(Y,Z) + Y(Z,X) - Z(X,Y)$$

$$= < [X,Y], Z > + < [X,Z], Y > + < [Y,Z], X > + 2 < Z, \nabla_X Y >.$$

Solving for $< Z, \nabla_X Y >$ we get:

$$< Z, \nabla_X Y >= \frac{1}{2}(X(Y,Z) + Y(Z,X) - Z(X,Y)$$
$$- < [X,Y], Z > - < [X,Z], Y > \ - < [Y,Z], X >).$$

The equation above will allow us to calculate Γ^i_{jk} for ∇ in terms of the metric g. Thus, if ∇ exists, then it's unique. To prove that such a connection exists one starts with the last equation and shows it satisfies $\nabla g = 0$ and $[X,Y] = \nabla_X Y - \nabla_Y X$ (the two required conditions).

Proposition: Let (M, g) be a smooth manifold. Then, over a coordinate patch $U \subseteq M$ with local coordinates $(x^1, \dots, x^n)$ the Christoffel symbols of the Levi-Civita connection are given by

$$\Gamma^i_{jk} = \sum_{l=1}^{n} \frac{1}{2} g^{il} \left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right)$$

where g^{ij} are the entries to the inverse matrix (g_{kl}).

Proof: Let $X = \partial_i,\ Y = \partial_j,\ Z = \partial_k$, then by the last equation in the previous proposition we can write:

$$\begin{aligned} < \partial_k, \nabla_{\partial_i} \partial_j > = \frac{1}{2} (&\nabla_{\partial_i} < \partial_j, \partial_k > + \nabla_{\partial_j} < \partial_k, \partial_i > \\ &- \nabla_{\partial_k} < \partial_i, \partial_j > - < [\partial_i, \partial_j], \partial_k > \\ &- < [\partial_i, \partial_k], \partial_j > \ - < [\partial_j, \partial_k], \partial_i >). \end{aligned}$$

Notice that $[\partial_i, \partial_j] = \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} = 0$ since by the smoothness mixed partial derivatives are equal.

$$\begin{aligned} &< \partial_k, \sum_{l=1}^{n} \Gamma^l_{ij}\, \partial_l > \\ &\qquad = \frac{1}{2} \left(\nabla_{\partial_i} < \partial_j, \partial_k > + \nabla_{\partial_j} < \partial_k, \partial_i > - \nabla_{\partial_k} < \partial_i, \partial_j > \right). \end{aligned}$$

$< X, Y >$ is a smooth function on M so $\nabla_{\partial_i} < \partial_j, \partial_k >$ is the directional derivative of $< \partial_j, \partial_k >$ in the direction of ∂_i. Thus:

$$\nabla_{\partial_i} < \partial_j, \partial_k >= \frac{\partial}{\partial x^i}(g_{jk})$$

$$\sum_{l=1}^{n} \Gamma_{ij}^{l}\, g_{kl} \;= \frac{1}{2}\left(\frac{\partial}{\partial x^i}(g_{jk}) + \frac{\partial}{\partial x^j}(g_{ki}) - \frac{\partial}{\partial x^k}(g_{ij})\right).$$

Now multiply through by g^{kt} since $g_{kl}\, g^{kt} = \delta_l^t$.

$$\sum_{k=1}^{n}\sum_{l=1}^{n} \Gamma_{ij}^{l}\, g_{kl} g^{kt} = \frac{1}{2}\sum_{k=1}^{n} g^{kt}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

$$\Gamma_{ij}^{t} = \frac{1}{2}\sum_{k=1}^{n} g^{kt}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right).$$

By renaming indices we get:

$$\Gamma_{jk}^{i} = \sum_{l=1}^{n} \frac{1}{2} g^{il}\left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right).$$