Riemannian Metrics – Lengths and Volumes

Now that we have a metric on a Riemannian manifold, we can investigate how to measure lengths of curves and "volumes" of regions on manifolds (note: "volume" means length for a curve, surface area for a surface, volume for a 3-dimensional region, and n-dimensional volume for an n-dimensional region).

We know from second year calculus that the length of a curve in $\mathbb{R}^3$ (using the standard metric on $\mathbb{R}^3$) is given by:

$$l(\gamma) = \int_a^b \|\gamma'(t)\| \, dt$$

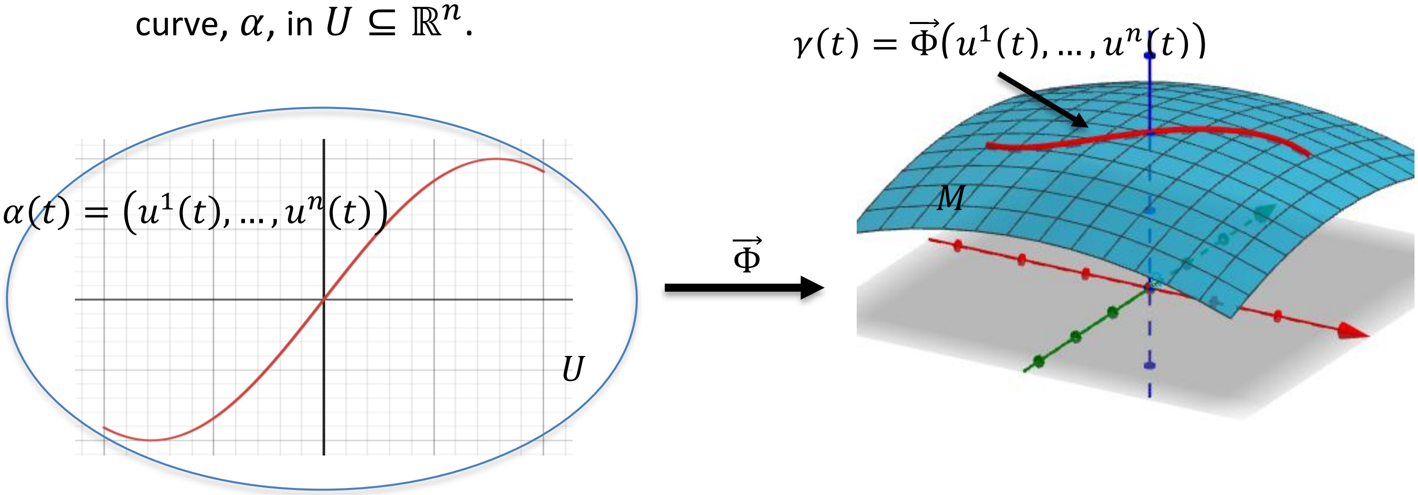
where the curve, γ, is given by:

$$\gamma(t) = < x(t), y(t), z(t) >; \qquad a \leq t \leq b.$$

We are going to generalize this definition to apply to curves on n-dimensional manifolds with a Riemannian metric, g. Suppose we can parametrize an n-dimensional manifold, M, by:

$$\vec{\Phi}: U \subseteq \mathbb{R}^n \to M \subseteq \mathbb{R}^k.$$

We can think of any curve, γ, that lays on M as the image under $\vec{\Phi}$ of a curve, α, in $U \subseteq \mathbb{R}^n$.



So $\gamma(t) = \vec{\Phi}\big(u^1(t), \dots, u^n(t)\big)$, where $\alpha(t) = \big(u^1(t), \dots, u^n(t)\big)$.

By the Chain Rule:

$$\gamma'(t) = \vec{\Phi}_{u^1}\frac{du^1}{dt} + \cdots + \vec{\Phi}_{u^n}\frac{du^n}{dt}$$

So we can write:

$$\begin{aligned}\|\gamma'(t)\|^2 &= \gamma'(t)\cdot\gamma'(t)\\ &= \left(\vec{\Phi}_{u^1}\frac{du^1}{dt} + \cdots + \vec{\Phi}_{u^n}\frac{du^n}{dt}\right)\cdot\left(\vec{\Phi}_{u^1}\frac{du^1}{dt} + \cdots + \vec{\Phi}_{u^n}\frac{du^n}{dt}\right)\\ &= \sum_{i,j=1}^{n} g_{ij}\frac{du^i}{dt}\frac{du^j}{dt} = <\gamma'(t),\gamma'(t)>\end{aligned}$$

where $<\ ,\ >$ is the inner product from the Riemannian metric g.

Notice that here we have used the metric induced by the parametrization $\vec{\Phi}$ of M. However, we might have a Riemannian metric that doesn't come from the parametrization $\vec{\Phi}$, but the following definition still holds.

Def. We define the **length of a curve**, γ, on a Riemannian manifold, (M, g), by:

$$l(\gamma) = \int_a^b \sqrt{<\gamma'(t),\gamma'(t)>}\,dt = \int_a^b \sqrt{\sum_{i,j=1}^{n} g_{ij}\frac{du^i}{dt}\frac{du^j}{dt}}\,dt.$$

Ex. Consider the portion of the great circle on the upper unit hemisphere starting at $(0, 0, 1)$ and ending at $(1, 0, 0)$. Find the length using the metric induced by:

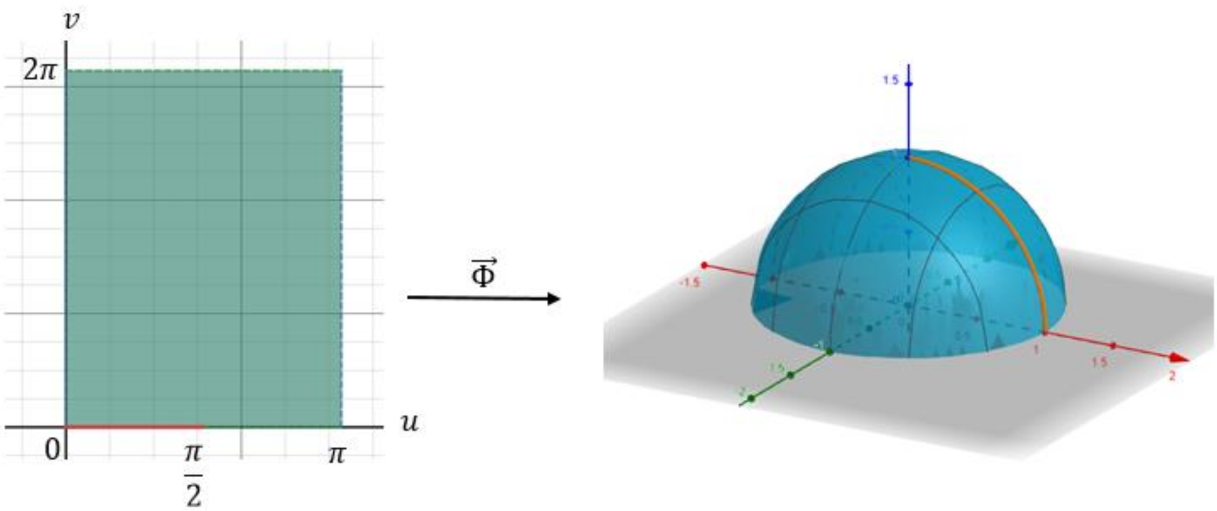
a) $\vec{\Phi}(u, v) = \big((\cos v)\sin u\,, (\sin v)\sin u\,, (\cos u)\big)$

$$0 \le u \le \frac{\pi}{2}, \quad 0 \le v \le 2\pi.$$

b) $\vec{\Psi}(u, v) = \left(u, v, \sqrt{1 - u^2 - v^2}\right); \quad u^2 + v^2 \le 1.$

$a)$ $\vec{\Phi}(u, v) = \big((\cos v)\sin u\,, (\sin v)\sin u\,, (\cos u)\big);$

$$0 \le u \le \frac{\pi}{2}, \quad 0 \le v \le 2\pi.$$



$$\vec{\Phi}_u = ((\cos v)\cos u\,, (\sin v)\cos u\,, -\sin u)$$

$$\vec{\Phi}_v = (-(\sin v)\sin u\,, (\cos v)\sin u\,, 0)$$

$$g_{11} = \vec{\Phi}_u \cdot \vec{\Phi}_u = 1 \qquad\qquad g_{12} = g_{21} = \vec{\Phi}_u \cdot \vec{\Phi}_v = 0$$

$$g_{22} = \vec{\Phi}_v \cdot \vec{\Phi}_v = \sin^2 u$$

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 u \end{pmatrix}.$$

The portion of the great circle we want is the image under $\overrightarrow{\Phi}$ of
$\alpha(t) = (t, 0); \quad 0 \le t \le \frac{\pi}{2}$.

i.e. $u(t) = t, \qquad v(t) = 0$
$u'(t) = 1, \qquad v'(t) = 0$

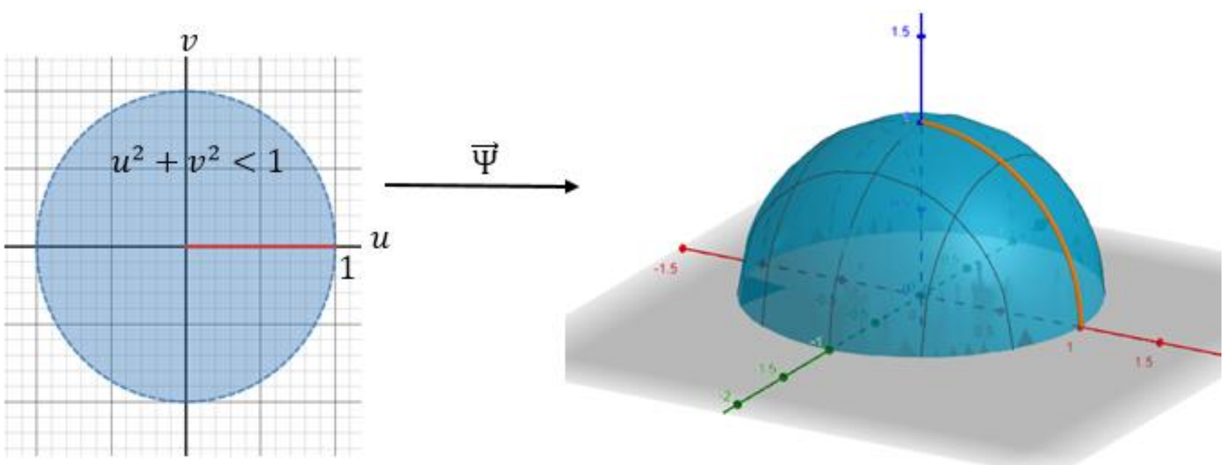
$$L = \int_0^{\frac{\pi}{2}} (g_{11}(u')^2 + 2g_{12}(u')(v') + g_{22}(v')^2)^{\frac{1}{2}}\, dt$$

$$= \int_0^{\frac{\pi}{2}} 1\, dt = \frac{\pi}{2}.$$

b. We saw earlier that the metric induced on S^2 by
$\overrightarrow{\Psi}(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right)$ is:

$$\bar{g} = \begin{pmatrix} \frac{1-v^2}{1-u^2-v^2} & \frac{uv}{1-u^2-v^2} \\ \frac{uv}{1-u^2-v^2} & \frac{1-u^2}{1-u^2-v^2} \end{pmatrix}$$

The portion of the great circle we want is the image of the line segment starting at $u = 0, v = 0$ and ending at $u = 1, v = 0$.



$\alpha(t) = (t, 0); \qquad 0 \le t \le 1$
$u(t) = t, \qquad v(t) = 0$
$u'(t) = 1, \qquad v'(t) = 0$

$$L = \int_0^1 (\bar{g}_{11}(u')^2 + 2\bar{g}_{12}(u')(v') + \bar{g}_{22}(v')^2)^{\frac{1}{2}}\, dt$$

$$= \int_0^1 \left(\frac{1}{1-t^2}\right)^{\frac{1}{2}} dt = \sin^{-1}(t)|_0^1 = \frac{\pi}{2}.$$

Any metric induced by a smooth parametrization of S_+^2 should give the same length of any curve on S_+^2. However, if we take a metric induced by a parametrization and alter it, the resulting length could change.

Ex. Let's take the metric induced on S_+^2 by :

$$\vec{\Phi}(u, v) = \big((\cos v)\sin u\,, (\sin v)\sin u\,, \cos u\big)$$

and alter it as in a. and b. below, to see what happens to the length of the portion of the great circle from $(0, 0, 1)$ to $(1, 0, 0)$.

a) Let $\tilde{g} = 2(g_{ij}) = 2\begin{pmatrix} 1 & 0 \\ 0 & \sin^2 u \end{pmatrix}$

b) Let $\bar{G} = \frac{1}{(1+u^2+v^2)^2}\begin{pmatrix} 1 & 0 \\ 0 & \sin^2 u \end{pmatrix}$.

The parametrization of the curve, γ, is still $\alpha(t) = (t, 0);\quad 0 \leq t \leq \frac{\pi}{2}$

$$u(t) = t, \qquad v(t) = 0$$
$$u'(t) = 1, \qquad v'(t) = 0.$$

a) So we can write:

$$L = \int_0^{\frac{\pi}{2}} (\tilde{g}_{11}(u')^2 + 2\tilde{g}_{12}(u')(v') + \tilde{g}_{22}(v')^2)^{\frac{1}{2}}\, dt$$
$$= \int_0^{\frac{\pi}{2}} (2)^{\frac{1}{2}}\, dt = \frac{\sqrt{2}\pi}{2}.$$

b) Here we get:

$$L = \int_0^{\frac{\pi}{2}} (\bar{G}_{11}(u')^2 + 2\bar{G}_{12}(u')(v') + \bar{G}_{22}(v')^2)^{\frac{1}{2}}\, dt$$
$$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{(1+t^2)^2}\right)^{\frac{1}{2}} dt = \int_0^{\frac{\pi}{2}} \frac{1}{1+t^2}\, dt$$
$$= \tan^{-1}(t)\Big|_0^{\frac{\pi}{2}} = \tan^{-1}\left(\frac{\pi}{2}\right).$$

In second year calculus, we learn that if a surface in $\mathbb{R}^3, S$, is parametrized by $\vec{\Phi}: U \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$, then the surface area of S is given by:

$$Area(S) = \iint_U \left\|\vec{\Phi}_u \times \vec{\Phi}_v\right\| du\, dv.$$

Through a messy calculation, it can be shown that if g is the induced metric from $\vec{\Phi}$ (i.e. $g_{ij} = \vec{\Phi}_{x^i} \cdot \vec{\Phi}_{x^j}$), then $\left\|\vec{\Phi}_u \times \vec{\Phi}_v\right\| = \sqrt{\det g}$ so

$$Area(S) = \iint_U \sqrt{\det g}\ du\, dv.$$

We can generalize this formula to any smooth surface in $\mathbb{R}^n$ (not just $\mathbb{R}^3$) and any Riemannian metric, g (not just metrics that are induced by parametrization).

In fact, the length of a curve formula is actually a special case of the "area" formula. We can parametrize a curve on an n-dimensional manifold by:

$$\vec{\Phi}(t) = \left(u^1(t), \dots, u^n(t)\right)$$

$$\vec{\Phi}_t = \left((u^1)', \dots, (u^n)'\right).$$

Since we only have one dimension, the metric tensor, g, is a 1×1 matrix:

$$g = \vec{\Phi}_t \cdot \vec{\Phi}_t = \left(\frac{du^1}{dt}\right)^2 + \cdots + \left(\frac{du^n}{dt}\right)^2.$$

And since g is a 1×1 matrix, $g = \det g$. Thus:

$$\sqrt{\det g} = \sqrt{\left(\frac{du^1}{dt}\right)^2 + \cdots + \left(\frac{du^n}{dt}\right)^2} = \left\|\vec{\Phi}'(t)\right\|.$$

Here $\vec{\Phi}'(t) = \gamma'(t)$ and so we can write:

$$\int_U \sqrt{\det g}\, dt = \int_U \|\gamma'(t)\|\, dt \qquad \text{where } U = [a, b].$$

Thus, the "area" formula is actually a generalization of the "length" formula. In fact, we can generalize the "area" formula to be an n-dimensional volume formula (where $n = 1$ is length, $n = 2$ is surface area).

Def. If g is a Riemannian metric on an n-dimensional manifold, M, then we define the $\boldsymbol{n}$**-dimensional volume** of M to be:

$$Vol(M) = \int_M \sqrt{\det g}\, dx^1 \cdots dx^n$$

where $x^1, \cdots, x^n$ are local coordinates. Here we are assuming that we can parametrize all of M with one set of coordinates.

Ex. Find the surface area of the torus $T^2 \subseteq \mathbb{R}^4$ parametrized by:

$$\vec{\Phi}(u, v) = (\cos u\,, \sin u\,, \cos v\,, \sin v); \quad (u, v) \in [0, 2\pi]^2.$$

$$\vec{\Phi}_u = (-\sin u\,, \cos u\,, 0, 0)$$
$$\vec{\Phi}_v = (0, 0, -\sin v\,, \cos v)$$

$$g_{11} = \vec{\Phi}_u \cdot \vec{\Phi}_u = 1$$
$$g_{12} = g_{21} = \vec{\Phi}_u \cdot \vec{\Phi}_v = 0$$
$$g_{22} = \vec{\Phi}_v \cdot \vec{\Phi}_v = 1$$

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\det g = 1.$$

$$S.Area = \int_0^{2\pi} \int_0^{2\pi} \sqrt{1}\ du\, dv = \int_0^{2\pi} u \Big|_0^{2\pi} dv = \int_0^{2\pi} 2\pi\, dv = 4\pi^2.$$

Ex. Find the 3-dimensional volume of the 3-dimensional torus $T = S^1 \times S^1 \times S^1 \subseteq \mathbb{R}^6$ given by:

$$\vec{\Phi}(u^1, u^2, u^3) = (\cos u^1, \sin u^1, \cos u^2, \sin u^2, \cos u^3, \sin u^3);$$
$$(u^1, u^2, u^3) \in [0, 2\pi]^3.$$

We can see from the previous example:

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \qquad \det g = 1$$

$$Volume = \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \sqrt{1}\ du^1 du^2 du^3 = \int_0^{2\pi}\int_0^{2\pi} u^1\Big|_0^{2\pi} du^2 du^3$$
$$= \int_0^{2\pi}\int_0^{2\pi} 2\pi\ du^2 du^3 = \int_0^{2\pi} 4\pi^2\, du^3 = 8\pi^3.$$

Ex. Find the area of the unit sphere in $\mathbb{R}^3$ using the parametrization:

$$\vec{\Phi}(u, v) = \big((\cos v)\sin u, (\sin v)\sin u, \cos u\big);$$
$$0 \le u \le \pi, \quad 0 \le v \le 2\pi.$$

We saw earlier that:

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 u \end{pmatrix}; \qquad \det g = \sin^2 u$$

$$Area = \int_{v=0}^{2\pi}\int_{u=0}^{\pi} \sqrt{\sin^2 u}\ du\, dv = \int_{v=0}^{2\pi}\int_{u=0}^{\pi} \sin u\ du\, dv$$

$$= \int_{v=0}^{2\pi} -\cos u\Big|_0^{\pi} dv = \int_{v=0}^{2\pi} 2\, dv = 4\pi\,.$$

Ex. Find the area of the subset of the unit disk in $\mathbb{R}^2$ given by

$x^2 + y^2 \leq \frac{1}{2}$, if the metric is given by:

$$g = \frac{1}{1-x^2-y^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\det g = \frac{1}{(1-x^2-y^2)^2}.$$

$$Area = \iint_{x^2+y^2\leq\frac{1}{2}} \sqrt{\frac{1}{(1-x^2-y^2)^2}}\, dx\, dy \ .$$

Switch to polar coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$dx\, dy = r dr d\theta.$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{\frac{\sqrt{2}}{2}} \left(\frac{1}{1-r^2}\right) r\, dr\, d\theta = \int_0^{2\pi} -\frac{1}{2}\ln(1-r^2)\Big|_0^{\frac{\sqrt{2}}{2}} d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}\left(\ln\left(1-\frac{1}{2}\right) - \ln(1)\right) d\theta = \int_0^{2\pi} -\frac{1}{2}\ln\left(\frac{1}{2}\right) d\theta$$

$$= -\frac{1}{2}\ln\left(\frac{1}{2}\right)\theta\Big|_0^{2\pi} = -\pi\ln\left(\frac{1}{2}\right) = \pi\ln 2 \ .$$