Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$- HW Problems

1. Let $\bar{x}_1 = x_1\cos(x_2)$ and $\bar{x}_2 = x_1\sin(x_2)$. Suppose that $f: \mathbb{R}^2 \to \mathbb{R}$ is a smooth function of $\bar{x}_1$ and $\bar{x}_2$. Show that:

$$\left(\frac{\partial f}{\partial \bar{x}_1}\right)^2 + \left(\frac{\partial f}{\partial \bar{x}_2}\right)^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 + \frac{1}{x_1^2}\left(\frac{\partial f}{\partial x_2}\right)^2.$$

2. Let $z_1 = 2y_1 + 3y_2 \qquad y_1 = e^{x_1} + x_2$

$$z_2 = y_1 y_2^2 \qquad y_2 = e^{(x_1 - x_2)} + x_1.$$

Use the chain rule to find $\frac{\partial z_i}{\partial x_j}$ for $i, j = 1,2$ in term of x_1 and x_2. You don't need to simplify your answer.

3. Let $x = 2r - s$ and $y = r + 2s$. Let $U: \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Find $\frac{\partial^2 U}{\partial y \partial x}$ in terms of derivatives of only U with respect to r and s.

4. Suppose $\Phi: \mathbb{R}^2 \to \mathbb{R}^3$ by $\Phi(\mathrm{x}_1, x_2) = (u(x_1, x_2), v(x_1, x_2), w(x_1, x_2))$ is a smooth function. Furthermore, suppose that $\alpha: \mathbb{R} \to \mathbb{R}^2$ is a smooth curve given by $\alpha(t) = \big(f(t), g(t)\big)$. Using the chain rule find an expression for $\frac{d}{dt}(\Phi(\alpha(\mathrm{t}))$, in terms of u, v, w, f, and g (and/or their derivatives).

5. Let $f(x, y, z) = x^2yz^3$.

   a. Find the directional derivative of f at the point $(1,1,-1)$ in the direction of $\vec{w} =< -1,2,1 >$.

   b. Find the directional derivative of f at any point (x, y, z) in the direction of $\vec{w} =< -1,2,1 >$.

   Note: Your answer to a. should be a number while your answer to b. should be a non-constant function of x, y, z.