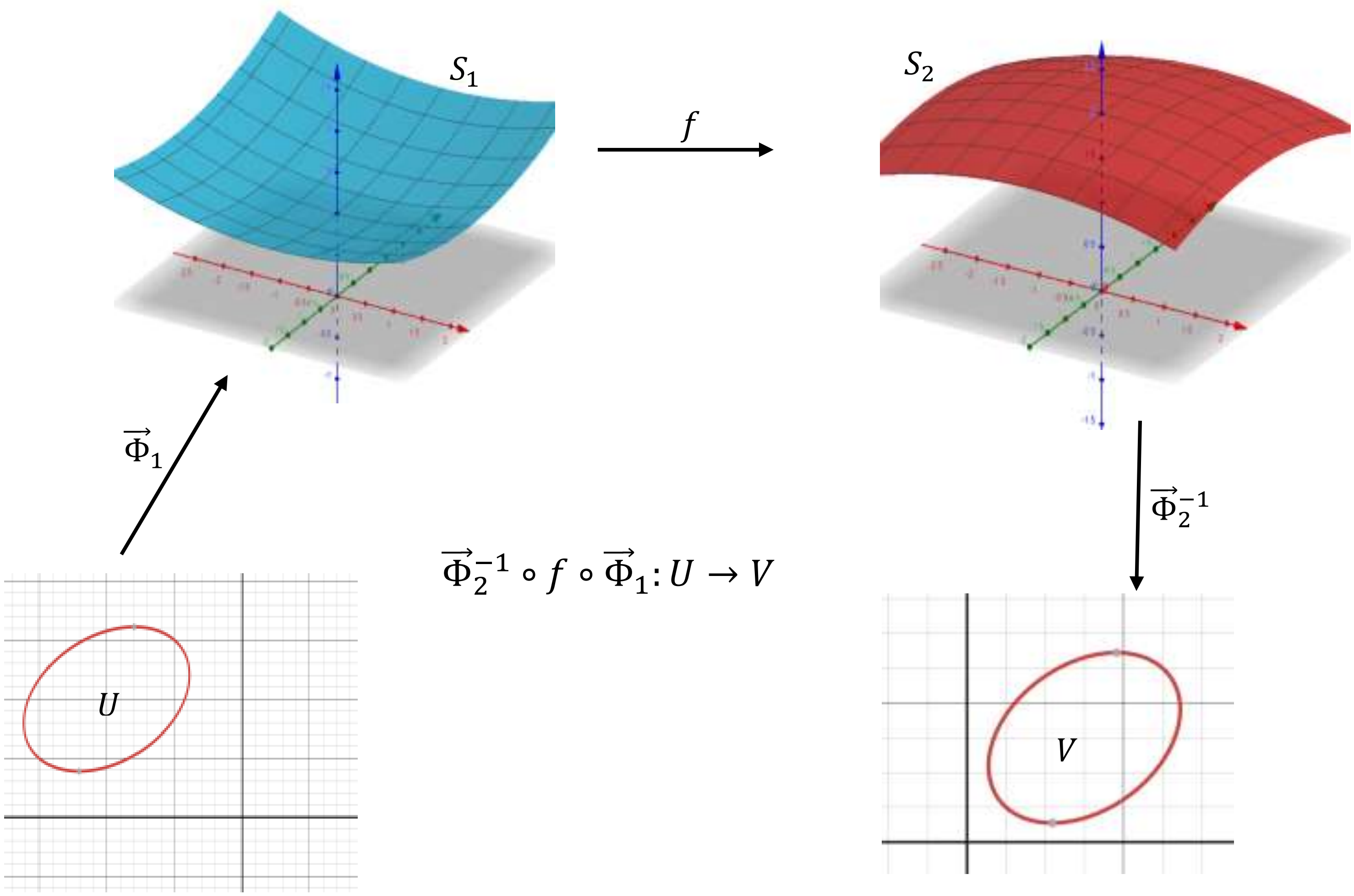
Smooth Maps, Tangent Planes and Derivatives

Given two smooth surfaces S_1, S_2 we want to define the notion of a smooth map $f: S_1 \to S_2$.

If we let $\vec{\Phi}_1: U \subseteq \mathbb{R}^2 \to S_1$ and $\vec{\Phi}_2: V \subseteq \mathbb{R}^2 \to S_2$ be surface patches (i.e., smooth, regular parametrizations) then,

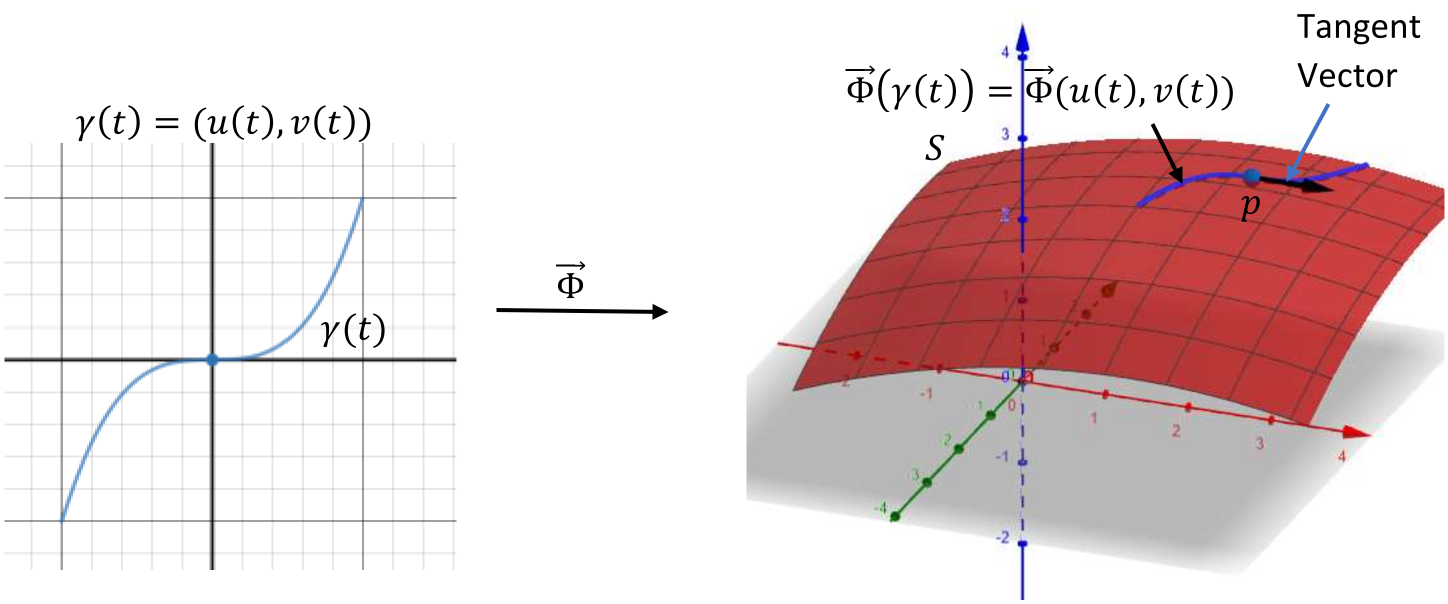


Since $U \subseteq \mathbb{R}^2$ and $V \subseteq \mathbb{R}^2$, we say f is **smooth** if $\vec{\Phi}_2^{-1} \circ f \circ \vec{\Phi}_1$ is smooth.

Def: If $f: S_1 \to S_2$ is smooth, 1-1, and onto, and $f^{-1}: S_2 \to S_1$ is smooth, we say f is a **diffeomorphism** and that S_1 and S_2 are **diffeomorphic**.

<u>Tangents and Derivatives</u>

Def: A **tangent vector** to a surface, S, at a point, $p \in S$, is the tangent vector at p of a curve in S passing through p. The **tangent space** (or **tangent plane**) of S at p is the set of all tangent vectors to S at p. We denote this by: T_pS.



If $\gamma(t) = \big(u(t), v(t)\big)$ is a smooth curve, then $\vec{\Phi}\big(u(t), v(t)\big)$ is a smooth curve on S. By the chain rule:

$$\frac{d}{dt}\vec{\Phi}\big(u(t), v(t)\big) = \vec{\Phi}_u\left(\frac{du}{dt}\right) + \vec{\Phi}_v\left(\frac{dv}{dt}\right)$$

is a tangent vector on S at $\vec{\Phi}\big(u(t), v(t)\big)$. In fact, all tangent vectors to S at $p \in S$ are spanned by:

$$\vec{\Phi}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \text{ and } \vec{\Phi}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

since $\vec{\Phi}$ is assumed to be regular (i.e. $\vec{\Phi}_u \times \vec{\Phi}_v \neq \vec{0}$).

Thus all linear combinations of $\vec{\Phi}_u$ and $\vec{\Phi}_v$, i.e., $\{a\vec{\Phi}_u + b\vec{\Phi}_v | a, b \in \mathbb{R}\}$ is the tangent plane of S at $p \in S$.

Ex. Let S be given by $z^2 = x^2 + y^2$, $z > 0$ (portion of a circular cone with $z > 0$). Find an equation of the tangent plane to S at $(0,1,1)$ i.e., $T_{(0,1,1)}S$.

We can parametrize S by:

$\vec{\Phi}(u, v) = (u\cos v, u\sin v, u)$; $u > 0$, $0 \le v \le 2\pi$.

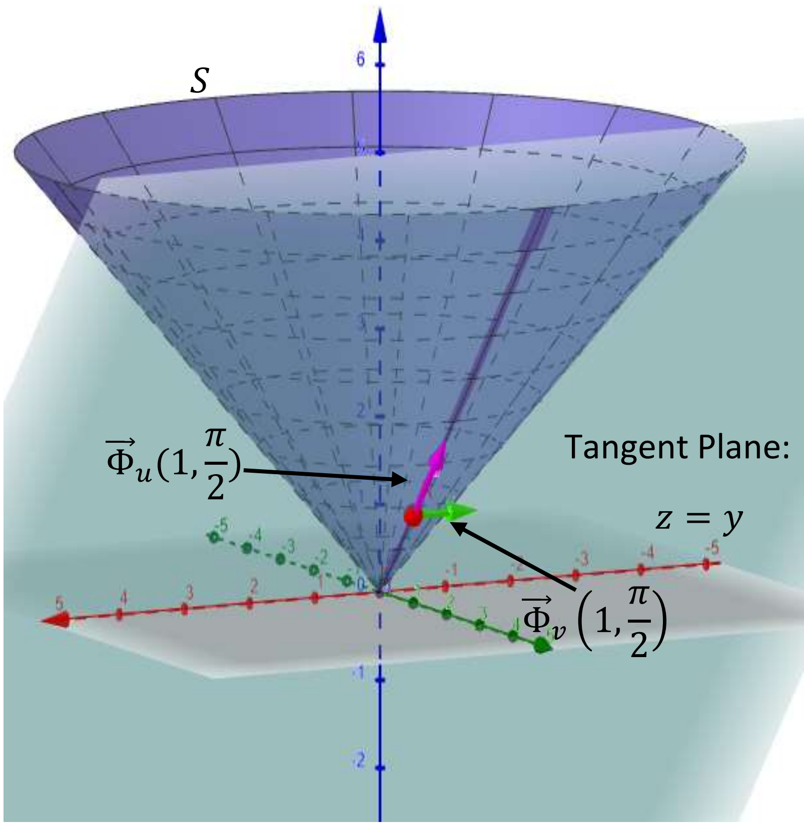
The point $(0,1,1)$ corresponds to:

$$0 = u\cos v$$

$$1 = u\sin v$$

$$1 = u$$

$$\Longrightarrow \quad u = 1,\ v = \frac{\pi}{2}.$$



We first find $\vec{\Phi}_u(1, \frac{\pi}{2})$ and $\vec{\Phi}_v(1, \frac{\pi}{2})$:

$\vec{\Phi}_u = (\cos v, \sin v, 1)$,

so $\vec{\Phi}_u(1, \frac{\pi}{2}) = (0, 1, 1)$

$\vec{\Phi}_v = (-u\sin v, u\cos v, 0)$,

so $\vec{\Phi}_v\left(1, \frac{\pi}{2}\right) = (-1, 0, 0)$.

To find the tangent plane at $u = 1, v = \frac{\pi}{2}$ we need to find a normal vector to the surface which we get from:

$$\vec{\Phi}_u\left(1, \frac{\pi}{2}\right) \times \vec{\Phi}_v\left(1, \frac{\pi}{2}\right).$$

$$\vec{\Phi}_u\left(1, \frac{\pi}{2}\right) \times \vec{\Phi}_v\left(1, \frac{\pi}{2}\right) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{vmatrix} = -\vec{\jmath} + \vec{k}.$$

Recall that if we have a normal vector $\vec{N} = (A, B, C)$ and a point, $q = (x_0, y_0, z_0)$, on a plane, then an equation for the plane is:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

In this case, $\vec{N} = (0, -1,1)$, $q = (0,1,1)$, so an equation for the tangent plane to S at $(0,1,1)$ is:

$$0(x - 0) - 1(y - 1) + 1(z - 1) = 0$$

or

$$-y + z = 0.$$

Or in vector form we can write the tangent plane as:

$$T_{(0,1,1)}S = \{(0,1,1) + s(0,1,1,) + t(-1,0,0); \quad s, t \in \mathbb{R}\}.$$

Given a smooth map between two surfaces

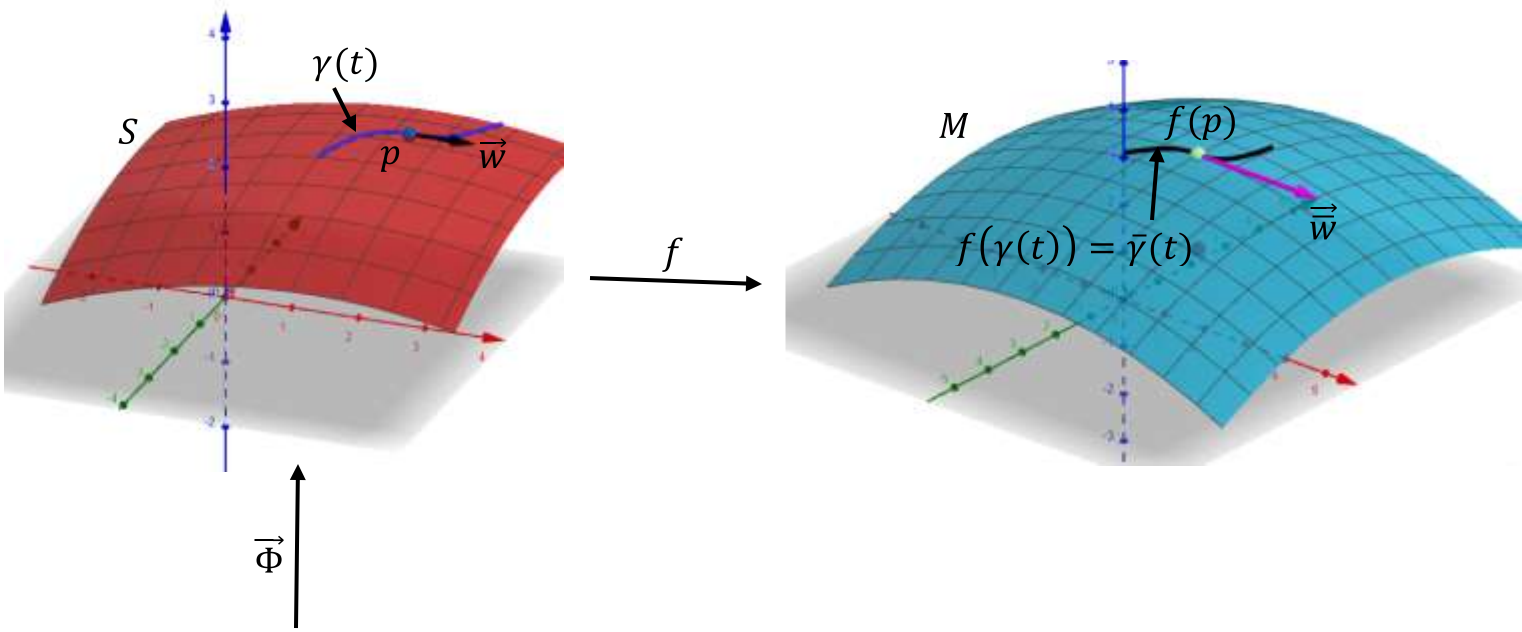
$$f: S \to M$$

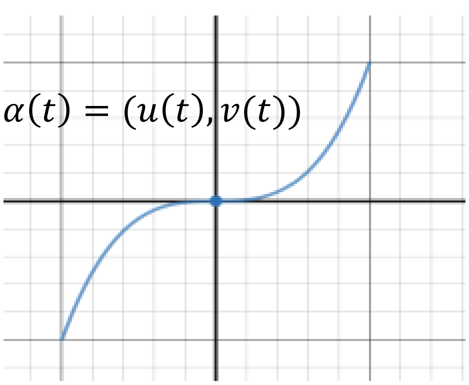
We want to define what we mean by the derivative of f (or differential of f) at $p \in S, D_p f$.

$$D_p f: T_p S \to T_{f(p)} M$$

by assigning any vector $\vec{w} \in T_p S$ to a vector in $T_{f(p)} M$. This is done as follows, given any vector, $\vec{w} \in T_p S$, there exists a curve, γ, in S passing through p, i.e., $\gamma(t_0) = p \in S$, such that $\gamma'(t_0) = \vec{w}$. Then $\bar{\gamma}(t) = f\big(\gamma(t)\big)$ is a curve in M passing through $f(p)$ at $t = t_0$. So $\vec{\bar{w}} = \bar{\gamma}'(t_0) \in T_{f(p)} M$. We define $D_p f$ by:

$$D_p f(\vec{w}) = \vec{\bar{w}} \in T_{f(p)} M$$





One can check that this definition does not depend on which curve γ in S one chooses such that $\gamma(t_0) = p \in S$ and $\gamma'(t_0) = \vec{w}$.

How do we calculate $D_p f$?

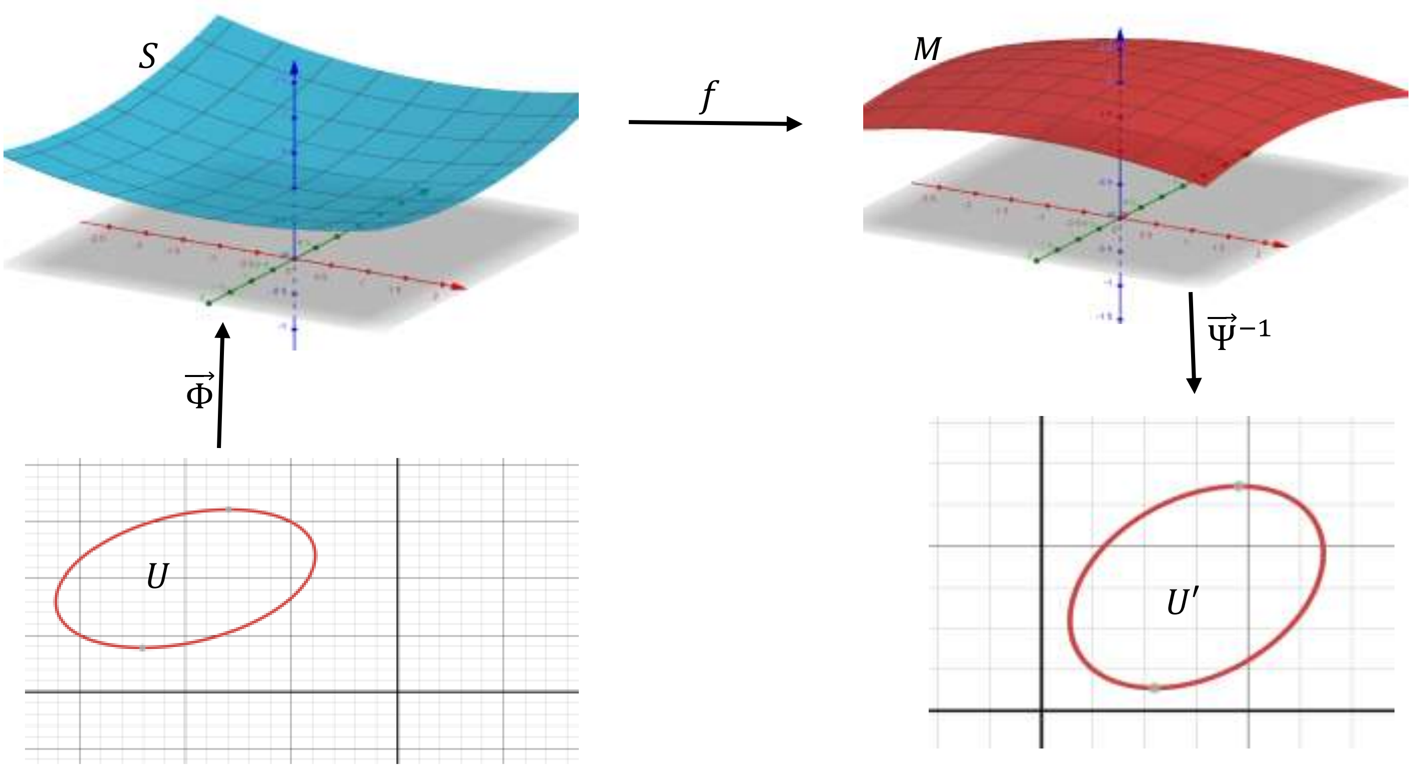
Suppose $\vec{\Phi}: U \to \mathbb{R}^3$ is a surface patch for S containing p and $\vec{\Psi}: U' \to \mathbb{R}^3$ is a surface patch for M containing $f(p)$. Since U and U' are open sets in $\mathbb{R}^2$ there is a smooth map

$$U \to U'$$

$$(u, v) \to (u', v') = \big(\alpha(u, v), \beta(u, v)\big)$$

such that:

$$\vec{\Psi}^{-1} \circ f\left(\vec{\Phi}(u, v)\right) = \big(\alpha(u, v), \beta(u, v)\big).$$



Then:

$$D_p f = \begin{pmatrix} \alpha_u & \alpha_v \\ \beta_u & \beta_v \end{pmatrix}$$

with respect to the basis $\{\vec{\Phi}_u, \vec{\Phi}_v\}$ for T_pS and $\{\vec{\Psi}_{u'}, \vec{\Psi}_{v'}\}$ for $T_{f(p)}M$.

This is the Jacobian matrix for the map:

$$(u, v) \to \big(\alpha(u, v), \beta(u, v)\big).$$

Ex. Suppose $S = M =$ upper hemisphere of the unit sphere. Parametrized by:

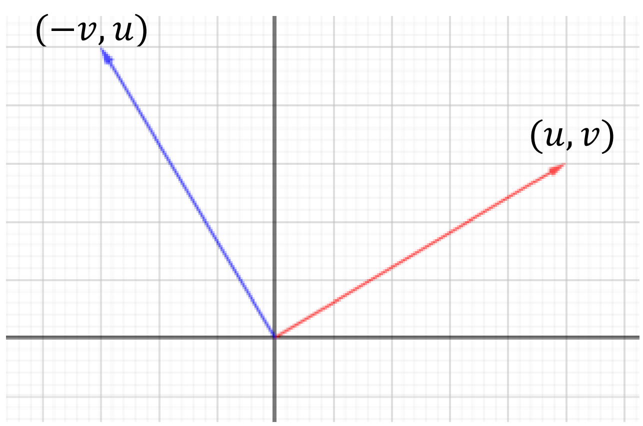
$$\vec{\Phi}(u, v) = \left(u, v, \sqrt{1 - u^2 - v^2}\right) \qquad \text{for } S$$

$$\vec{\Psi}(u', v') = \left(u', v', \sqrt{1 - (u')^2 - (v')^2}\right) \qquad \text{for } M.$$

Suppose $f: S \to S$ by a rotation of $\frac{\pi}{2}$ about the z axis. Find $D_p f$.

Rotating about the z axis is equivalent to rotating $U = \{(u, v) \in \mathbb{R}^2 |\, u^2 + v^2 < 1\}$ about the origin.

The map $(u, v) \to (-v, u)$ does this:



Thus, we have:

$$(u, v) \overset{\vec{\Phi}}{\to} \left(u, v, \sqrt{1 - u^2 - v^2}\right) \overset{f}{\to} \left(-v, u, \sqrt{1 - u^2 - v^2}\right)$$

$$f\left(\vec{\Phi}(u, v)\right) = \left(-v, u, \sqrt{1 - u^2 - v^2}\right).$$

Since $\vec{\Psi}(u',v') = \left(u', v', \sqrt{1-(u')^2-(v')^2}\right)$, we have:

$\vec{\Psi}^{-1}(x,y,z) = (x,y)$ and

$$\vec{\Psi}^{-1} \circ f\left(\vec{\Phi}(u,v)\right) = (-v,u) = \left(\alpha(u,v), \beta(u,v)\right).$$

Thus we have:

$$\alpha(u,v) = -v, \qquad \beta(u,v) = u.$$

$$D_p f = \begin{pmatrix} \alpha_u & \alpha_v \\ \beta_u & \beta_v \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Ex. Find $D_p f$ if S is the surface in $\mathbb{R}^3$ given by:

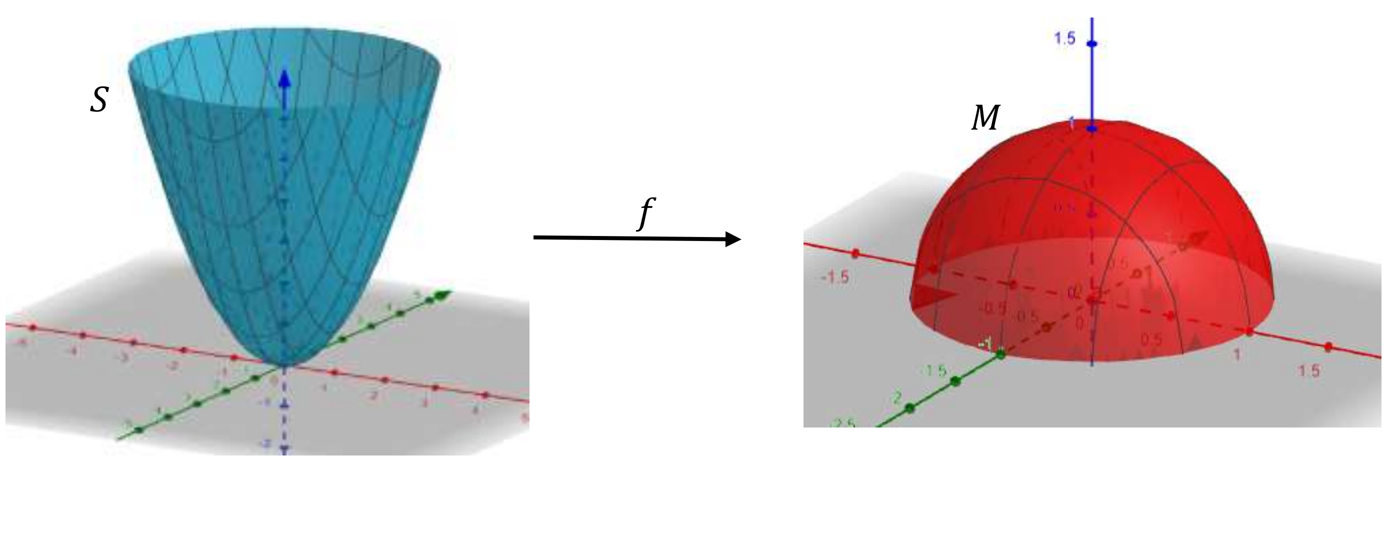
$$\vec{\Phi}(u,v) = (u,\ v,\ u^2+v^2); \qquad u,v \in \mathbb{R},$$

and M be the upper hemisphere given by:

$$\vec{\Psi}(\mathrm{u}',\mathrm{v}') = \left(\mathrm{u}', \mathrm{v}', \sqrt{1-(\mathrm{u}')^2-(\mathrm{v}')^2}\right); \quad (\mathrm{u}')^2+(\mathrm{v}')^2 < 1$$

and $f: S \to M$ by:

$$f(u,v,u^2+v^2) = \left(\frac{-2u}{\sqrt{1+4u^2+4v^2}}, \frac{-2v}{\sqrt{1+4u^2+4v^2}}, \frac{1}{\sqrt{1+4u^2+4v^2}}\right).$$



$\vec{\Psi}^{-1}(x, y, z) = (x, y)$ and

$f\left(\vec{\Phi}(u, v)\right) = \left(\frac{-2u}{\sqrt{1+4u^2+4v^2}}, \frac{-2v}{\sqrt{1+4u^2+4v^2}}, \frac{1}{\sqrt{1+4u^2+4v^2}}\right)$, so

$$\vec{\Psi}^{-1} \circ f\left(\vec{\Phi}(u, v)\right) = \left(\frac{-2u}{\sqrt{1+4u^2+4v^2}}, \frac{-2v}{\sqrt{1+4u^2+4v^2}}\right).$$

$$\alpha(u, v) = \frac{-2u}{\sqrt{1+4u^2+4v^2}}, \qquad \beta(u, v) = \frac{-2v}{\sqrt{1+4u^2+4v^2}}.$$

$$D_p f = \begin{pmatrix} \alpha_u & \alpha_v \\ \beta_u & \beta_v \end{pmatrix}$$

$$\alpha_u = \frac{-2(1+4v^2)}{(1+4u^2+4v^2)^{\frac{3}{2}}} \qquad \alpha_v = \frac{8uv}{(1+4u^2+4v^2)^{\frac{3}{2}}}$$

$$\beta_u = \frac{8uv}{(1+4u^2+4v^2)^{\frac{3}{2}}} \qquad \beta_v = \frac{-2(1+4u^2)}{(1+4u^2+4v^2)^{\frac{3}{2}}}$$

So we have:

$$D_p f = -\frac{2}{(1+4u^2+4v^2)^{\frac{3}{2}}} \begin{pmatrix} (1+4v^2) & -4uv \\ -4uv & 1+4u^2 \end{pmatrix}.$$

Ex. In the previous example, take the point $p = (1,2,5) \in S$ (i.e., $u = 1, v = 2$) and consider the vector $\vec{w} \in T_pS$ given by $\vec{w} = 3\vec{\Phi}_u(1,2) + \vec{\Phi}_v(1,2)$. Find $D_pf(\vec{w})$ with respect to the basis for $T_{f(p)}M$ given by $\vec{\Psi}_{u'}(\mathrm{f(p)})$, $\vec{\Psi}_{v'}(\mathrm{f(p)})$ as well as the standard basis for $\mathbb{R}^3$.

The matrix $D_pf = \begin{pmatrix} \alpha_u & \alpha_v \\ \beta_u & \beta_v \end{pmatrix}$ is a map from T_pS to $T_{f(p)}M$ with respect to the basis $\{\vec{\Phi}_u, \vec{\Phi}_v\}$ for T_pS and $\{\vec{\Psi}_{u'}, \vec{\Psi}_{v'}\}$ for T_pM.

In this case $p = (1,2,5) \in S$, i.e., $u = 1,\ v = 2$ and

$$f(u, v, u^2 + v^2) = \left(\frac{-2u}{\sqrt{1+4u^2+4v^2}}, \frac{-2v}{\sqrt{1+4u^2+4v^2}}, \frac{1}{\sqrt{1+4u^2+4v^2}}\right),$$

thus $f(p) = f(1,2,5) = \left(-\frac{2}{\sqrt{21}}, -\frac{4}{\sqrt{21}}, \frac{1}{\sqrt{21}}\right) \in M$.

For any point $p \in S$ we have from the previous example:

$$D_pf = -\frac{2}{(1+4u^2+4v^2)^{\frac{3}{2}}}\begin{pmatrix} (1+4v^2) & -4uv \\ -4uv & 1+4u^2 \end{pmatrix}.$$

Thus when $p = (1,2,5) \in S$, i.e., $u = 1, v = 2$ we have:

$$D_{(1,2,5)}f = -\frac{2}{(21)^{\frac{3}{2}}}\begin{pmatrix} 17 & -8 \\ -8 & 5 \end{pmatrix}.$$

With respect to the basis $\{\vec{\Phi}_u(1,2), \vec{\Phi}_v(1,2)\}$ we have

$\vec{w} =< 3,1 >$. When we apply $D_p f$ to $\vec{w}$ we get:

$$D_{(1,2,5)} f(\vec{w}) = -\frac{2}{(21)^{\frac{3}{2}}}\begin{pmatrix} 17 & -8 \\ -8 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = -\frac{2}{(21)^{\frac{3}{2}}}\begin{pmatrix} 43 \\ -19 \end{pmatrix}.$$

The vector $-\frac{2}{(21)^{\frac{3}{2}}} < 43, -19 >$ is already written with respect to the

basis $\{\vec{\Psi}_{u'}(f(p)),\ \vec{\Psi}_{v'}(f(p))\}$.

How do we write $-\frac{2}{(21)^{\frac{3}{2}}} < 43, -19 >$ with respect to the standard

basis for $\mathbb{R}^3$?

To do that we need to write $\vec{\Psi}_{u'}(f(p)), \vec{\Psi}_{v'}(f(p))$ in terms of the

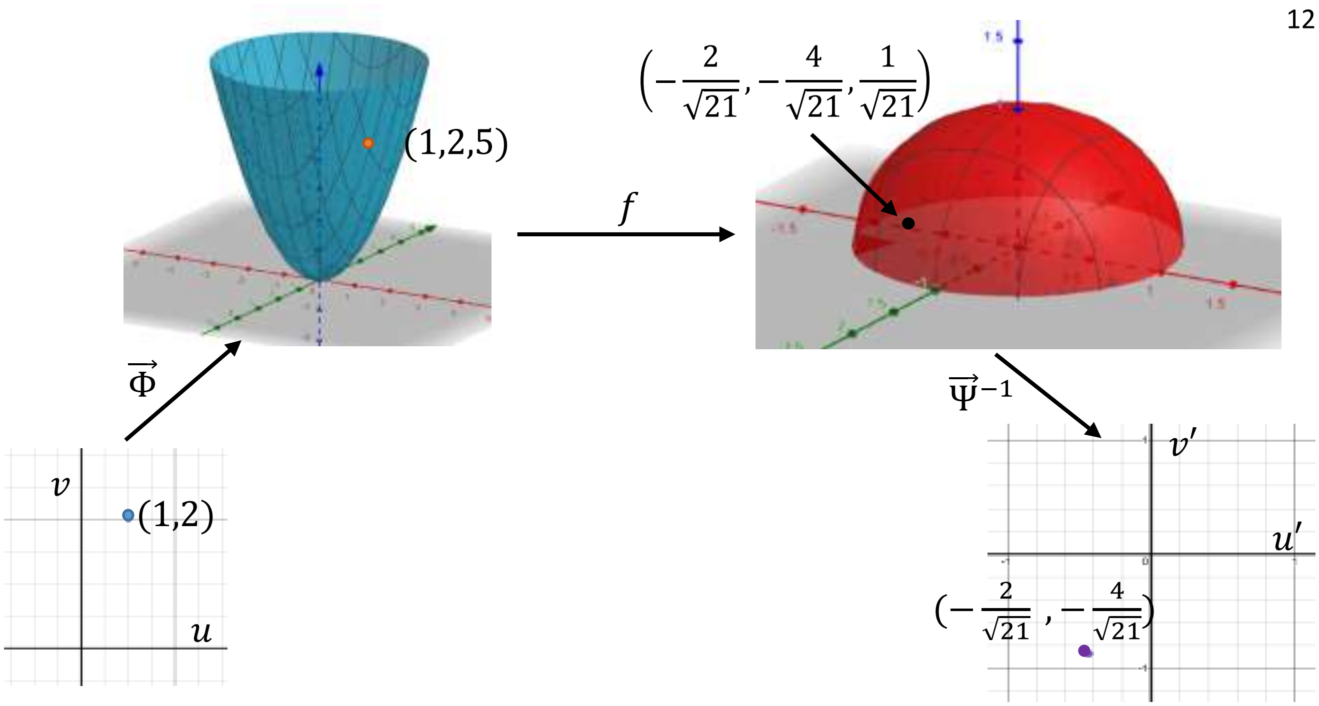
standard basis in $\mathbb{R}^3$.

From the previous problem we see that:

$$(u', v') = \vec{\Psi}^{-1} \circ f\left(\vec{\Phi}(u,v)\right) = \left(\frac{-2u}{\sqrt{1+4u^2+4v^2}}, \frac{-2v}{\sqrt{1+4u^2+4v^2}}\right).$$

So $u' = \frac{-2u}{\sqrt{1+4u^2+4v^2}}$ and $v' = \frac{-2v}{\sqrt{1+4u^2+4v^2}}$.

When $u = 1,\ v = 2,$ we have $u' = -\frac{2}{\sqrt{21}},\ \ v' = -\frac{4}{\sqrt{21}}$.



$\vec{\Psi}(u',v') = \left(u',v',\sqrt{1-(u')^2-(v')^2}\right)$, so

$$\vec{\Psi}_{u'} = (1,0,-\frac{u'}{\sqrt{1-(u')^2-(v')^2}}) \Rightarrow \vec{\Psi}_{u'}\left(-\frac{2}{\sqrt{21}},-\frac{4}{\sqrt{21}}\right) =< 1,0,2 >.$$

$$\vec{\Psi}_{v'} = (0,1,-\frac{v'}{\sqrt{1-(u')^2-(v')^2}}) \Rightarrow \vec{\Psi}_{v'}\left(-\frac{2}{\sqrt{21}},-\frac{4}{\sqrt{21}}\right) =< 0,1,4 >.$$

So now we can find $(D_{(1,2,5)}f)(\vec{w})$ in the standard basis for $\mathbb{R}^3$.

$$(D_{(1,2,5)}f)(\vec{w}) = -\frac{2}{(21)^{\frac{3}{2}}} < 43,-19 >$$

$$= -\frac{2}{(21)^{\frac{3}{2}}}\left(43\vec{\Psi}_{u'}\left(-\frac{2}{\sqrt{21}},-\frac{4}{\sqrt{21}}\right) - 19\vec{\Psi}_{v'}\left(-\frac{2}{\sqrt{21}},-\frac{4}{\sqrt{21}}\right)\right)$$

$$= -\frac{2}{(21)^{\frac{3}{2}}}(43 < 1,0,2 > -19 < 0,1,4 >)$$

$$= -\frac{2}{(21)^{\frac{3}{2}}} < 43,-19,10 >.$$