Smooth Surfaces

Def: Let U be an open set in $\mathbb{R}^m$, we say:

$$\vec{f}: U \to \mathbb{R}^n \text{ by:}$$

$$\vec{f}(x_1, \dots, x_m) = \big(f_1(x_1, \dots, x_m), f_2(x_1, x_2, \dots, x_m), \dots, f_n(x_1, \dots, x_m)\big)$$

is **smooth** if $f_i(x_1, \dots, x_m)$ has continuous partial derivatives of all orders for $i = 1, \dots, n$.

So if $\vec{\Phi}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$, then $\vec{\Phi}$ is smooth if:

$$\vec{\Phi}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right); \qquad \vec{\Phi}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

$$\vec{\Phi}_{uu} = \left(\frac{\partial^2 x}{\partial u^2}, \frac{\partial^2 y}{\partial u^2}, \frac{\partial^2 z}{\partial u^2}\right); \quad \vec{\Phi}_{uv} = \left(\frac{\partial^2 x}{\partial u \partial v}, \frac{\partial^2 y}{\partial u \partial v}, \frac{\partial^2 z}{\partial u \partial v}\right), \dots$$

has continuous partial derivatives of all orders.

Def. Let $U \subseteq \mathbb{R}^2$ be an open set. A **surface patch** (i.e. a parametrization) $\vec{\Phi}: U \to \mathbb{R}^3$ of a surface, S, is called regular if it is smooth and the vectors:

$$\vec{\Phi}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \text{ and } \vec{\Phi}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

are linearly independent at all points $(u,v) \in U$. This is equivalent to saying: $\vec{\Phi}_u \times \vec{\Phi}_v \neq 0$ for any point $(u,v) \in U$.

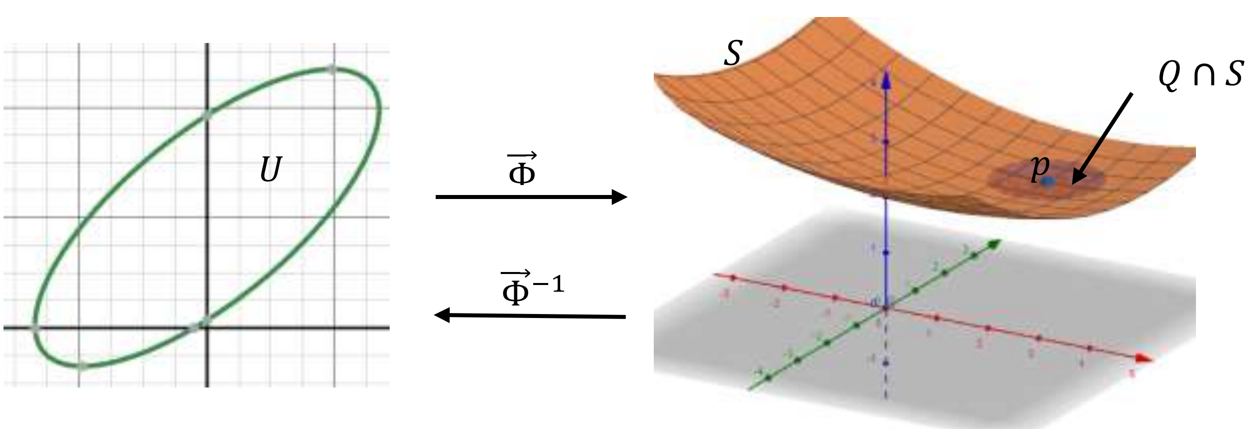
Def. If S is a surface, an **allowable surface patch** for S is a regular surface patch, $\vec{\Phi}: U \to \mathbb{R}^3$, such that $\vec{\Phi}$ and $\vec{\Phi}^{-1}$ are continuous (this is called a **homeomorphism**) from U to an open subset of S. A **smooth or regular surface** is a surface, S, such that for any point $p \in S$, there is an allowable patch, $\vec{\Phi}$, such that $p \in \vec{\Phi}(U)$. A collection, A, of allowable surface patches for a surface S such that every point of S is in the image of at least one patch in A is called an **atlas** for the smooth surface S.

Def. (Alternative Definition) A subset $S \subseteq \mathbb{R}^3$ is a **smooth or regular surface** if for each point $p \in S$ there exists an open set, $U \subseteq \mathbb{R}^2$, an open neighborhood Q of p in $\mathbb{R}^3$ and a surjective continuous function $\vec{\Phi}: U \to Q \cap S$ (surjective means every point $q \in Q \cap S$ has a point $p \in U$ such that $\vec{\Phi}(p) = q$) such that:

1) $\vec{\Phi}$ is smooth, i.e. if $\vec{\Phi}(u, v) = \big(x(u, v), y(u, v), z(u, v)\big)$ the $x(u, v), y(u, v)$ and $z(u, v)$ have continuous partial derivatives of all orders.

2) $\vec{\Phi}$ is a homeomorphism: $\vec{\Phi}$ is continuous and has an inverse function $\vec{\Phi}^{-1}: Q \cap S \to U$, which is also continuous.

3) $\vec{\Phi}_u \times \vec{\Phi}_v \neq \vec{0}$ for any $(u, v) \in U$.



Ex. Show that the unit cylinder $S = \{(x, y, z) \in \mathbb{R}^3 |\, x^2 + y^2 = 1\}$ is a smooth surface.

$$\vec{\Phi}(u, v) = (\cos u\,, \sin u\,, v); \quad 0 \le u \le 2\pi, \;\; v \in \mathbb{R}$$

is a parametrization of this cylinder, however the set:

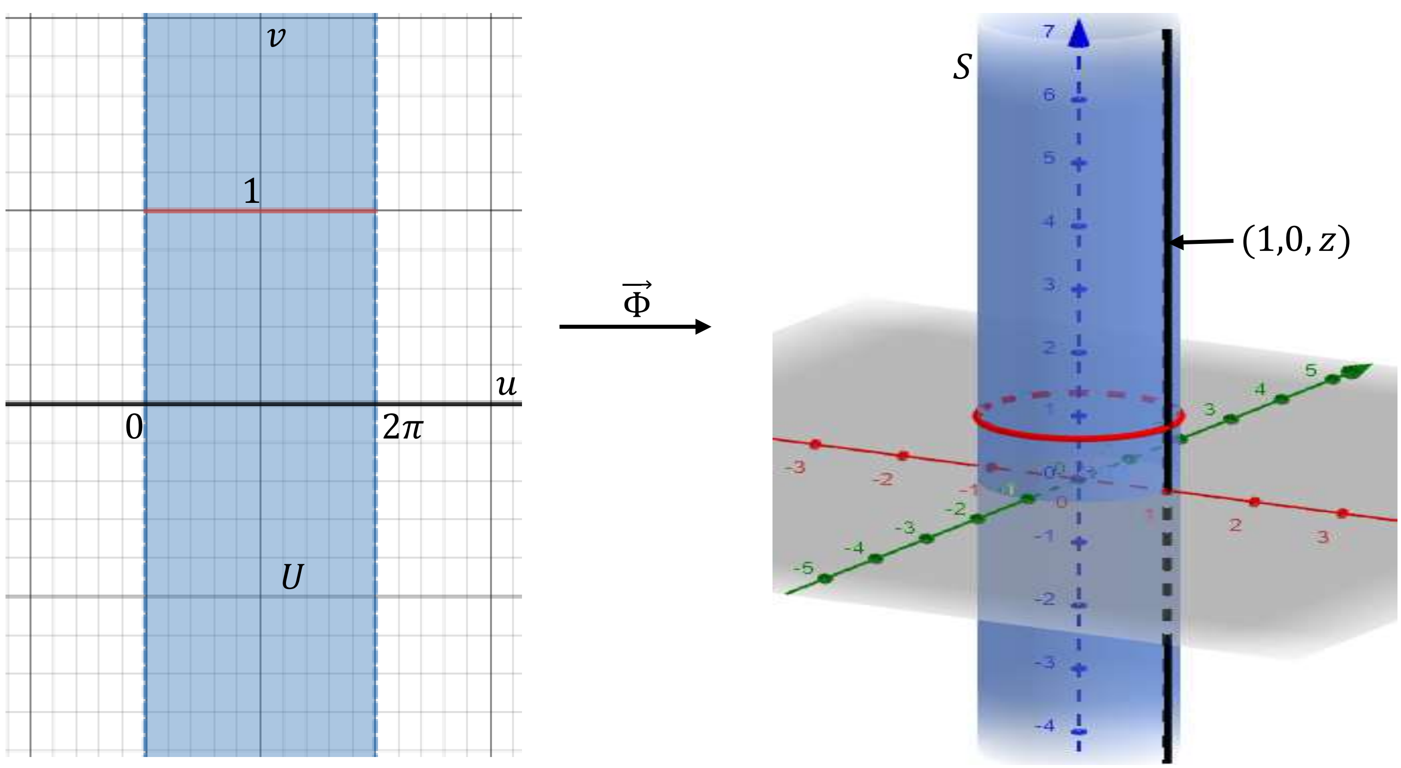
$$V = \{(u, v) \in \mathbb{R}^2 |\, 0 \le u \le 2\pi, v \in \mathbb{R}\}$$

is not an open set in $\mathbb{R}^2$.

So consider the open set:

$$U = \{(u, v) \in \mathbb{R}^2 |\, 0 < u < 2\pi, v \in \mathbb{R}\}.$$

U is open but $\vec{\Phi}(U)$ does not cover all of the surface S (the line $(1,0,z)$ is missing).



However, $\vec{\Phi}$ is $1-1$ from U into S, since if:

$$\vec{\Phi}(u_1, v_1) = \vec{\Phi}(u_2, v_2) \quad \text{then:}$$

$$(\cos u_1, \sin u_1, v_1) = (\cos u_2, \sin u_2, v_2)$$

This implies $v_1 = v_2$ and:

$$\cos u_1 = \cos u_2 \Longrightarrow \quad u_2 = u_1 \text{ or } 2\pi - u_1$$

$$\sin u_1 = \sin u_2 \Longrightarrow \quad u_2 = u_1 \text{ or } \pi - u_1.$$

So $u_1 = u_2$ and $\vec{\Phi}$ is $1-1$.

It's clear that $\vec{\Phi}$ is smooth. But we need to show that $\vec{\Phi}$ has a

a continuous inverse, $\vec{\Phi}^{-1}: \vec{\Phi}(U) \to U$. For $\{(x,y,z) \in \vec{\Phi}(U)\}$ define:

$$\begin{aligned}
\vec{\Phi}^{-1}(x,y,z) &= (Tan^{-1}\left(\frac{y}{x}\right), z) && \text{if } (x,y) \text{ is in the 1st quadrant} \\
&= (\frac{\pi}{2}, z) && \text{if } (x,y) = (0,1) \\
&= (\pi + Tan^{-1}\left(\frac{y}{x}\right), z) && \text{if } (x,y) \text{ is in the 2nd/3rd quadrant} \\
&= (\frac{3\pi}{2}, z) && \text{if } (x,y) = (0,-1) \\
&= (2\pi + Tan^{-1}\left(\frac{y}{x}\right), z) && \text{if } (x,y) \text{ is in the 4th quadrant.}
\end{aligned}$$

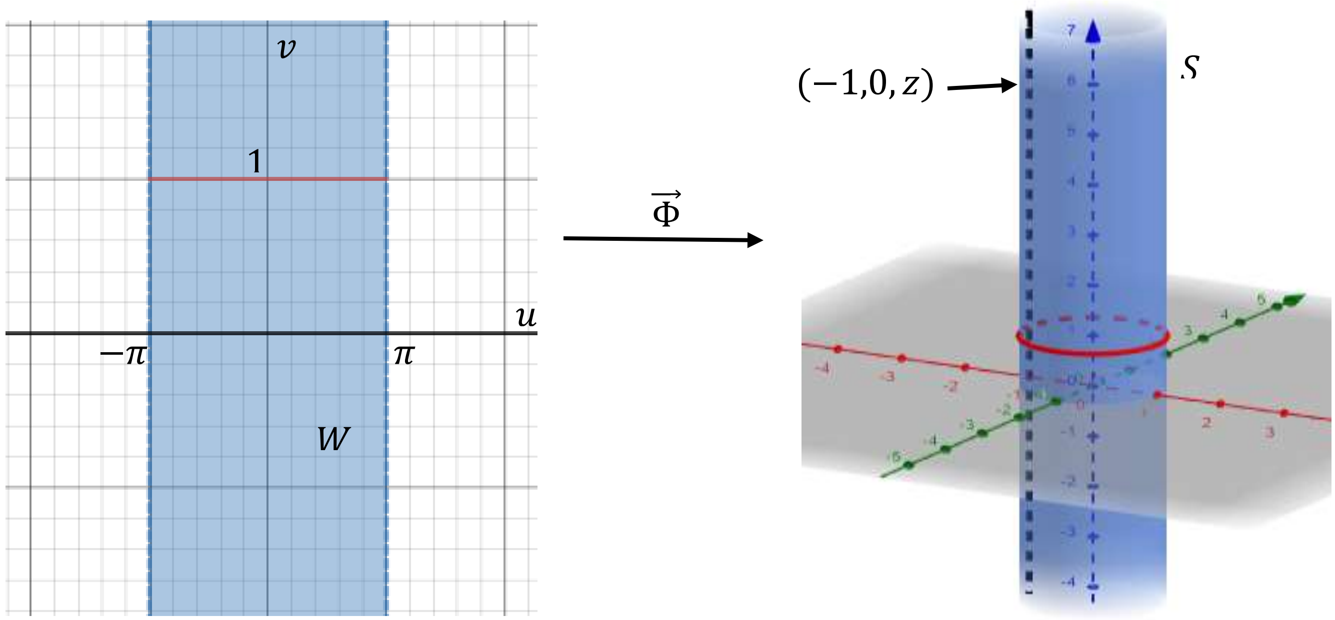
We need this messy inverse function because $-\frac{\pi}{2} < \tan^{-1} t < \frac{\pi}{2}$ and $\vec{\Phi}^{-1}$ maps $\vec{\Phi}(U)$ onto $U = (0, 2\pi) \times (-\infty, \infty)$.

It's not too difficult to show that $\vec{\Phi}^{-1}$ is continuous.

But we need another open set, W, so that $\vec{\Phi}: W \subseteq \mathbb{R}^2 \to S$ such that the points that we missed by $\vec{\Phi}: U \to S$ get covered by $\vec{\Phi}(W)$.

Here we can just take:

$$W = \{(u, v) \in \mathbb{R}^2 | -\pi < u < \pi, v \in \mathbb{R}\}.$$



By the same argument, $\vec{\Phi}$ is $1-1$ on W

Now we have:

$$\vec{\Phi}_u = (-\sin u, \cos u, 0)$$

$$\vec{\Phi}_v = (0,0,1).$$

$\vec{\Phi}_u$ and $\vec{\Phi}_v$ are linearly independent because one vector is not a multiple of the other. We can also see it by:

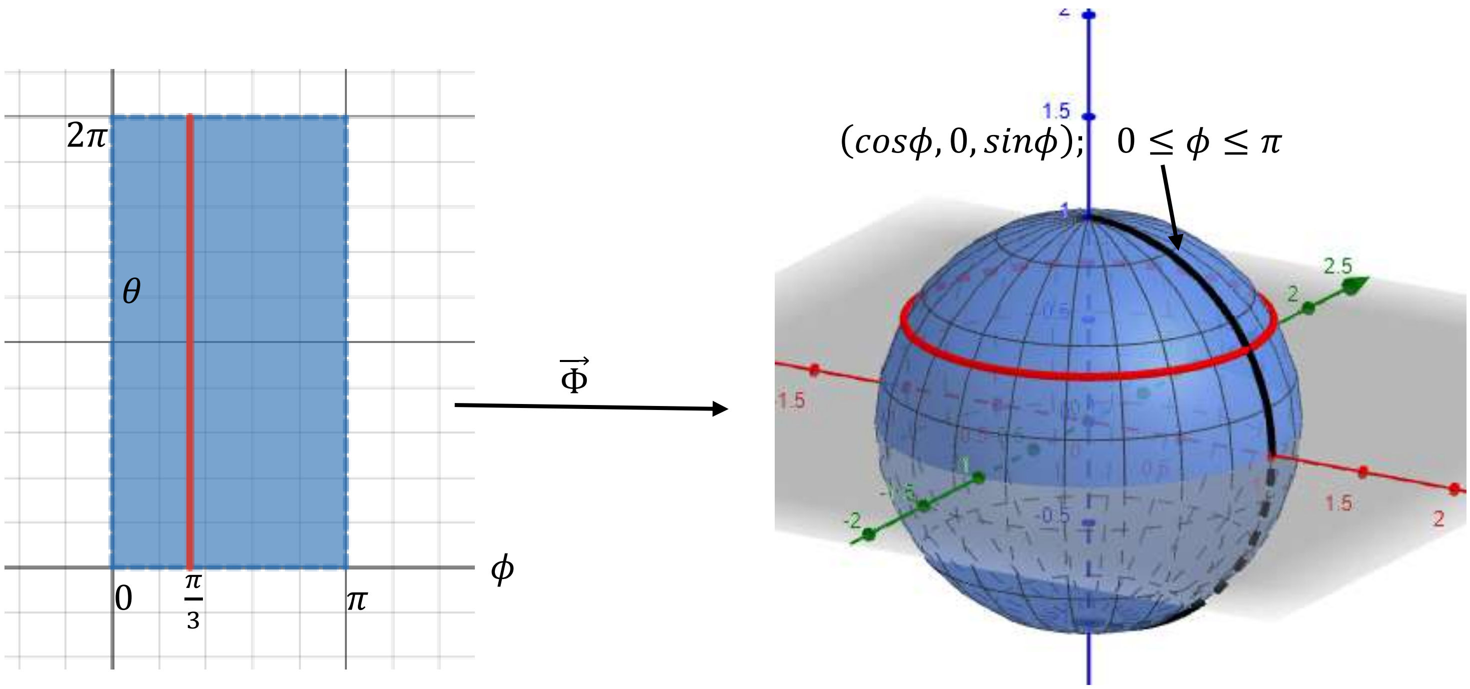
$$\vec{\Phi}_u \times \vec{\Phi}_v = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos u)\vec{\imath} + (\sin u)\vec{\jmath} \neq \vec{0}$$

So S is a smooth surface.

Ex. Show that the unit sphere $S = \{(x, y, z) |\ x^2 + y^2 + z^2 = 1\}$ is a smooth surface.

One smooth, regular parametrization of the unit sphere is given by:

$$\vec{\Phi}(\phi, \theta) = (\cos\theta\sin\phi\,, \sin\theta\sin\phi\,, \cos\phi);\ \ 0 < \phi < \pi, 0 < \theta < 2\pi.$$



In a manner similar to the previous example, it can be shown that

$\vec{\Phi}$ is $1-1$ and has a continuous inverse given by:

$$\vec{\Phi}^{-1}(x, y, z) = (\cos^{-1} z\,,\ Tan^{-1}\left(\frac{y}{x}\right)) \qquad \text{if } (x, y) \text{ is in the 1st quadrant}$$

$$= (\cos^{-1} z\,,\ \frac{\pi}{2}) \qquad \text{if } (x, y) = (0,1)$$

$$= (\cos^{-1} z\,,\ \pi + Tan^{-1}\left(\frac{y}{x}\right)) \qquad \text{if } (x, y) \text{ is in the 2nd/3rd quadrant}$$

$$= (\cos^{-1} z\,,\ \frac{3\pi}{2}) \qquad \text{if } (x, y) = (0, -1)$$

$$= (\cos^{-1} z\,,\ 2\pi + Tan^{-1}\left(\frac{y}{x}\right)) \qquad \text{if } (x, y) \text{ is in the 4th quadrant.}$$

To show $\vec{\Phi}_u \times \vec{\Phi}_v \neq \vec{0}$:

$$\vec{\Phi}_\phi = (\cos\theta\cos\phi\,, \sin\theta\cos\phi\,, -\sin\phi)$$

$$\vec{\Phi}_\theta = (-\sin\theta\sin\phi\,, \cos\theta\sin\phi\,, 0)$$

$$\vec{\Phi}_\phi \times \vec{\Phi}_\theta = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \\ -\sin\theta\sin\phi & \cos\theta\sin\phi & 0 \end{vmatrix}$$

$$= \cos\theta\sin^2\phi\,\vec{\imath} + \sin\theta\sin^2\phi\,\vec{\jmath} + (\cos^2\theta\sin\phi\cos\phi + \sin^2\theta\sin\phi\cos\phi)\,\vec{k}$$

$$\vec{\Phi}_\phi \times \vec{\Phi}_\theta = \cos\theta\sin^2\phi\,\vec{\imath} + \sin\theta\sin^2\phi\,\vec{\jmath} + \sin\phi\cos\phi\,\vec{k}.$$

We can see that $\vec{\Phi}_\phi \times \vec{\Phi}_\theta \neq 0$ by showing its length is never 0.

$$\left\|\vec{\Phi}_\phi \times \vec{\Phi}_\theta\right\| = \sqrt{\cos^2\theta\sin^4\phi + \sin^2\theta\sin^4\phi + \sin^2\phi\cos^2\phi}$$

$$= \sqrt{(\cos^2\theta + \sin^2\theta)\sin^4\phi + \sin^2\phi\cos^2\phi}$$

$$= \sqrt{\sin^4\phi + \sin^2\phi\,(\cos^2\phi)}$$

$$= \sqrt{\sin^2\phi\,(\sin^2\phi + \cos^2\phi)}$$

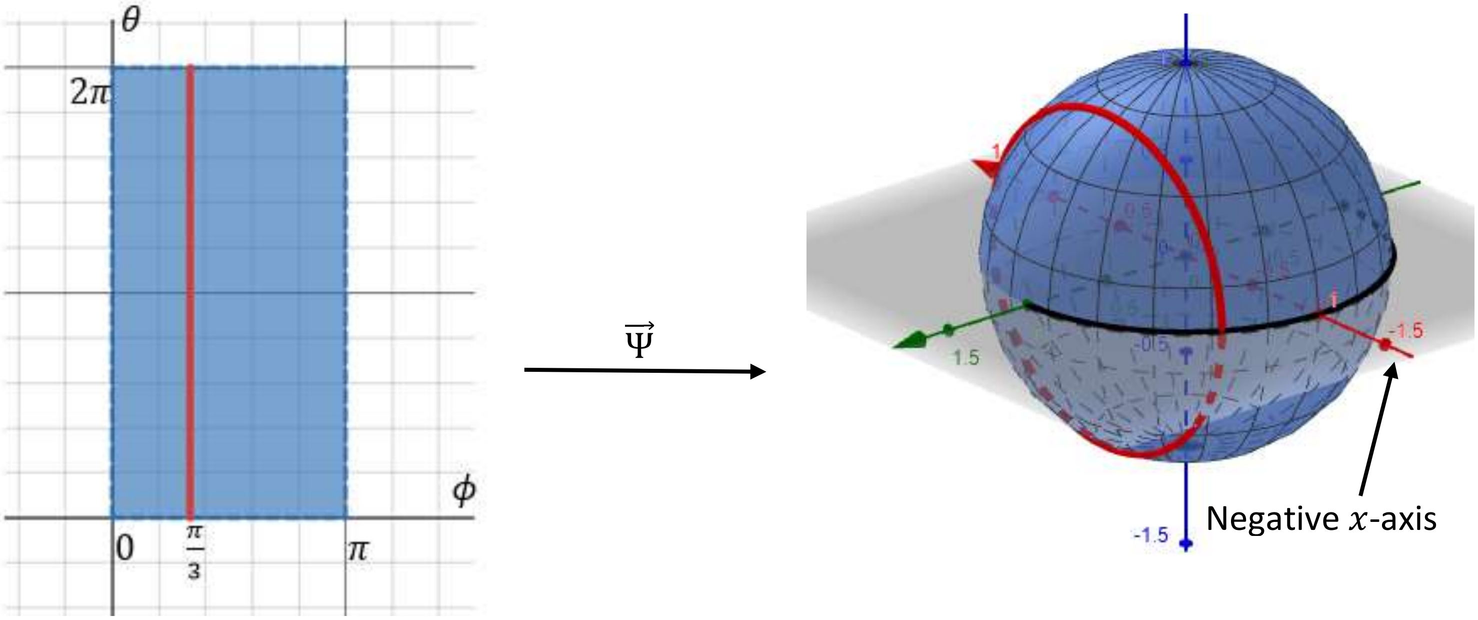
$$= |\sin\phi| > 0 \quad \text{since} \quad 0 < \phi < \pi.$$

However, $\vec{\Phi}$ does not map the set:

$$U = \{(\phi, \theta) \in \mathbb{R}^2 |\, 0 < \phi < \pi,\ 0 < \theta < 2\pi\}$$

onto the unit sphere. The semicircle on the sphere of points of the form $(x, 0, z),\ x \geq 0$ is missing (black semi-cirlce in previous picture).

So now let's consider the set: $H = \{(x, y, z) \in \mathbb{R}^3 |\, x \leq 0, z = 0\}$, and consider $W = S - H$ (W is open). W is the sphere with the semicircle $(x, y, 0),\ x \leq 0$ missing.



Now define: $\vec{\Psi}: U \to S - H = W$ by,

$$\vec{\Psi}(\phi, \theta) = (-\cos\theta \sin\phi\,, \cos\phi\,, -\sin\theta \sin\phi), 0 < \phi < \pi, 0 < \theta < 2\pi.$$

As with the map $\vec{\Phi}$, $\vec{\Psi}$ is smooth and a homeomorphism.

And we have:

$$\vec{\Psi}_\phi = (-\cos\theta\cos\phi\,, -\sin\phi\,, -\sin\theta\cos\phi)$$

$$\vec{\Psi}_\theta = (\sin\theta\sin\phi\,, 0, -\cos\theta\sin\phi)$$

$$\vec{\Psi}_\phi \times \vec{\Psi}_\theta = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -\cos\theta\cos\phi & -\sin\phi & -\sin\theta\cos\phi \\ \sin\theta\sin\phi & 0 & -\cos\theta\sin\phi \end{vmatrix}$$

$$= \cos\theta\sin^2\phi\,\vec{\imath} - (\cos^2\theta\cos\phi\sin\phi + \sin^2\theta\cos\phi\sin\phi)\vec{\jmath} + \sin\theta\sin^2\phi\,\vec{k}$$

$$= (\cos\theta\sin^2\phi)\vec{\imath} - (\sin\phi\cos\phi)\vec{\jmath} + \sin\theta\sin^2\phi\,\vec{k}.$$

As before, $\left\|\vec{\Psi}_\phi \times \vec{\Psi}_\theta\right\| = |\sin\phi| > 0$ since $0 < \phi < \pi$.

So $\left\|\vec{\Psi}_\phi \times \vec{\Psi}_\theta\right\| \neq \vec{0}$.

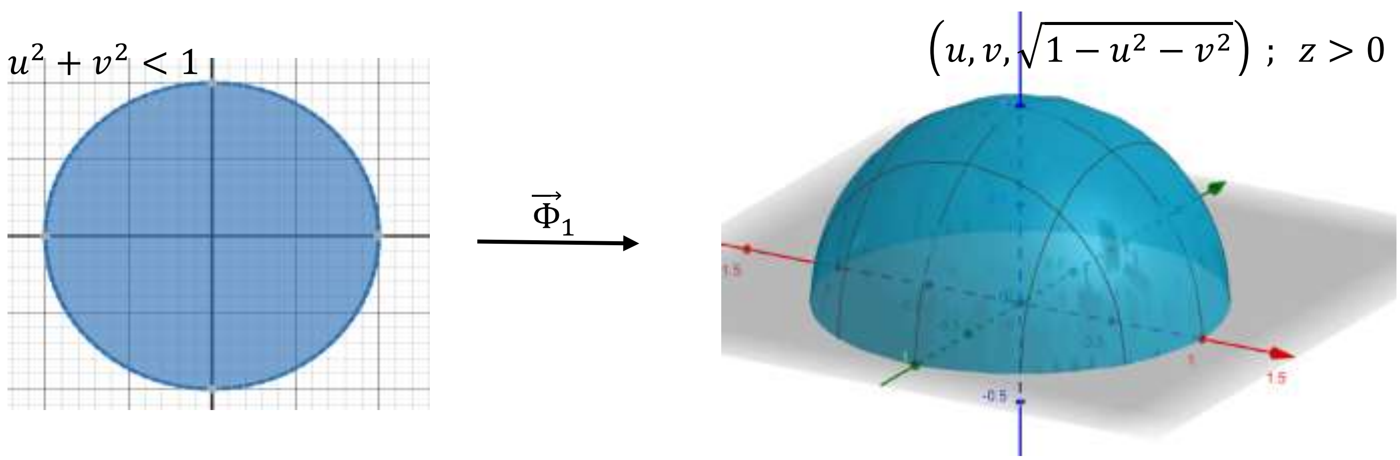
Since $\vec{\Phi}(U) \cup \vec{\Psi}(W) \supseteq S$ and $\vec{\Phi}$ and $\vec{\Psi}$ are homeomorphisms, S is a smooth surface.

This is not the only way to show that S is a smooth surface. For example, we could use 6 open patches (open sets) on the sphere. Each of the maps below map the set $U = \{(u,v) | u^2+v^2<1\}$ in $\mathbb{R}^2$ onto a different hemisphere:

$$\vec{\Phi}_1(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right); \qquad z>0$$
$$\vec{\Phi}_2(u,v) = \left(u, v, -\sqrt{1-u^2-v^2}\right); \qquad z<0$$
$$\vec{\Phi}_3(u,v) = \left(u, \sqrt{1-u^2-v^2}, v\right); \qquad y>0$$
$$\vec{\Phi}_4(u,v) = \left(u, -\sqrt{1-u^2-v^2}, v\right); \qquad y<0$$
$$\vec{\Phi}_5(u,v) = \left(\sqrt{1-u^2-v^2}, u, v\right); \qquad x>0$$
$$\vec{\Phi}_6(u,v) = \left(-\sqrt{1-u^2-v^2}, u, v\right); \qquad x<0$$

Each of these $\vec{\Phi}_i s$ has the property that it's a homeomorphism from the disk $u^2+v^2<1$ in $\mathbb{R}^2$ onto a hemisphere and it is regular.

We'll demonstrate this with $\vec{\Phi}_1(u,v)$.



1. $\vec{\Phi}_1$ is clearly smooth for $u^2 + v^2 < 1$.
2. $\vec{\Phi}_1$ is 1-1 since:

$$\vec{\Phi}_1(u_1, v_1) = \vec{\Phi}_1(u_2, v_2)$$

$$\left(u_1, v_1, \sqrt{1 - u_1^2 - v_1^2}\right) = \left(u_2, v_2, \sqrt{1 - u_2^2 - v_2^2}\right)$$

$$\Longrightarrow u_1 = u_2 \quad v_1 = v_2, \text{ so } \vec{\Phi}_1 \text{ is 1-1.}$$

3. $\vec{\Phi}_1^{-1}(x, y, z) = (x, y)$ is the continuous inverse of $\vec{\Phi}_1$.
4. $\left(\vec{\Phi}_1\right)_u \times \left(\vec{\Phi}_1\right)_v \neq 0$ since:

$$\left(\vec{\Phi}_1\right)_u = \left(1, 0, \frac{-u}{\sqrt{1-u^2-v^2}}\right) \qquad \left(\vec{\Phi}_1\right)_v = \left(0, 1, \frac{-v}{\sqrt{1-u^2-v^2}}\right)$$

$$\left(\vec{\Phi}_1\right)_u \times \left(\vec{\Phi}_1\right)_v = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 0 & \frac{-u}{\sqrt{1-u^2-v^2}} \\ 0 & 1 & \frac{-v}{\sqrt{1-u^2-v^2}} \end{vmatrix} = \frac{u}{\sqrt{1-u^2-v^2}}\vec{\imath} + \frac{v}{\sqrt{1-u^2-v^2}}\vec{\jmath} + \vec{k}$$

$$\left\|\left(\vec{\Phi}_1\right)_u \times \left(\vec{\Phi}_1\right)_v\right\| = \sqrt{\frac{u^2}{1-u^2-v^2} + \frac{v^2}{1-u^2-v^2} + 1} = \sqrt{\frac{1}{1-u^2-v^2}} \neq 0$$

So $\left(\vec{\Phi}_1\right)_u \times \left(\vec{\Phi}_1\right)_v \neq 0$.

Since $\bigcup_{i=1}^{6} \vec{\Phi}_i\,(U) \supseteq S$, S is a smooth surface.