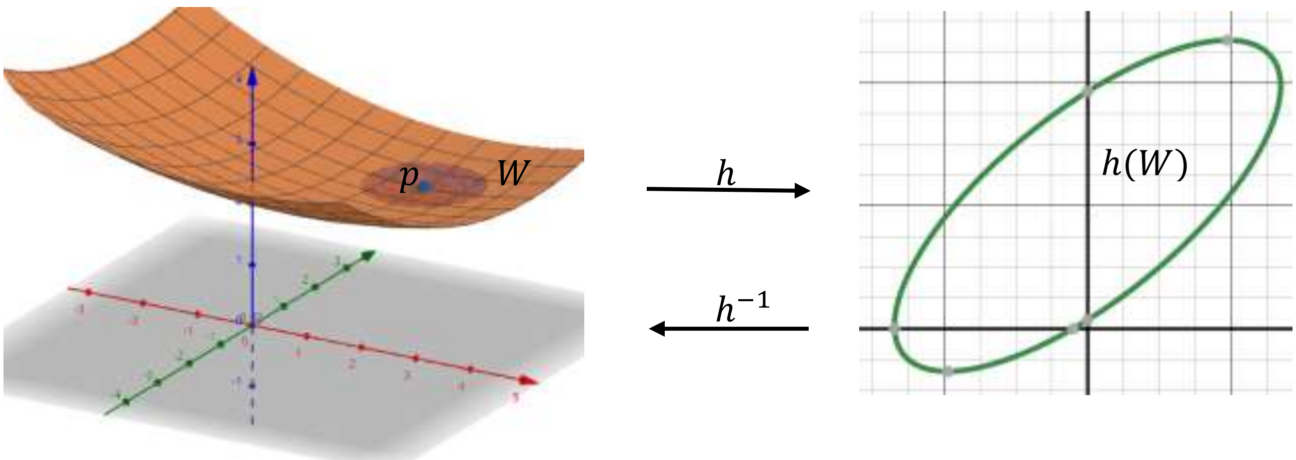
Surfaces in $\mathbb{R}^3$

A surface $S \subseteq \mathbb{R}^3$, is a subset of $\mathbb{R}^3$ where each point, p, of S has a neighborhood,W, which "looks like" (is homeomorphic to) an open set, $h(W)$, in $\mathbb{R}^2$.



Just as we had two natural ways to represent a curve in $\mathbb{R}^2$:

parametrically: $\quad \gamma(t) = \big(x(t), y(t)\big)$

and by an equation in x and y: $\quad f(x, y) = 0,$

we also have two natural ways to represent a surface in $\mathbb{R}^3$.

We can represent a surface parametrically by:

$$\overrightarrow{\Phi}: U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$$

$$\overrightarrow{\Phi}(u, v) = \big(x(u, v), y(u, v), z(u, v)\big).$$

A second way to represent a surface in $\mathbb{R}^3$ is by an equation:

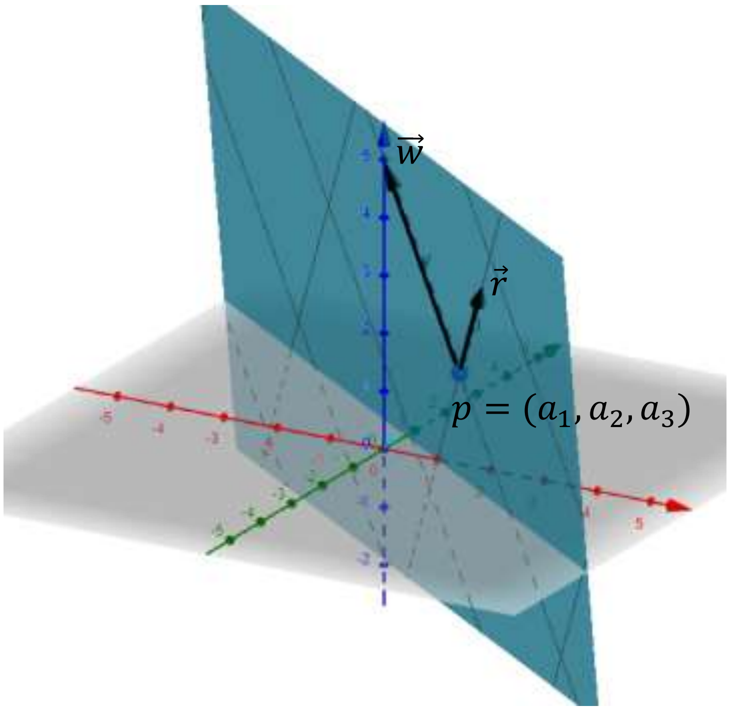
$$f(x, y, z) = 0.$$

Ex. We can represent a plane in $\mathbb{R}^3$ parametrically by:

$$\vec{\Phi}(u,v) = (a_1 + b_1 u + c_1 v, a_2 + b_2 u + c_2 v, a_3 + b_3 u + c_3 v)$$

where a_i, b_i, c_i are constants and

$\vec{r} = (b_1, b_2, b_3)$, $\vec{w} = (c_1, c_2, c_3)$ are non-parallel vectors.



Similarly, we can represent a plane in $\mathbb{R}^3$ with the equation:

$$Ax + By + Cz = D$$

where A, B, and C are constants.

If we let $\vec{r} \times \vec{w} = A\vec{i} + B\vec{j} + C\vec{k}$ then an equation of the plane is:

$$A(x - a_1) + B(y - a_2) + C(z - a_3) = 0.$$

Recall that if $y = f(x)$ or $x = g(y)$ are curves in $\mathbb{R}^2$, then we can always parametrize them by either

$$\gamma(t) = \big(t, f(t)\big) \quad \text{or} \quad \alpha(t) = (g(t), t).$$

Similarly, if we have a surface given as

$$z = f(x, y), \quad y = g(x, z), \quad \text{or} \quad x = h(y, z)$$

we can always represent it parametrically by:

$$\vec{\Phi}_1(u, v) = \big(u, v, f(u, v)\big), \qquad \vec{\Phi}_2(u, v) = (u, g(u, v), v),$$

$$\text{or} \qquad \vec{\Phi}_3(u, v) = (h(u, v), u, v).$$

Ex. Represent the elliptic paraboloid $x = y^2 + z^2$ parametrically.

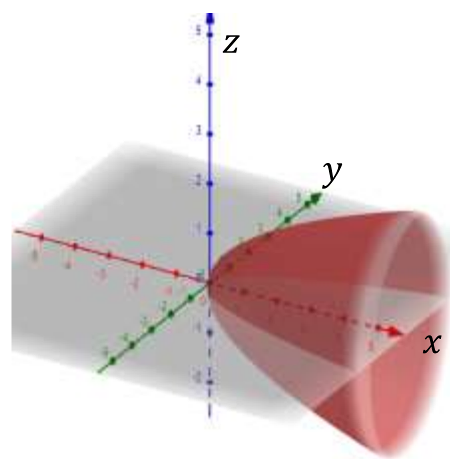
Here we have $x = f(y, z) = y^2 + z^2$.

So parametrically we have:

$$x = u^2 + v^2$$

$$y = u$$

$$z = v$$



or equivalently, $\vec{\Phi}(u, v) = (u^2 + v^2, u, v)$.

Ex. A circular cylinder is the set of points in $\mathbb{R}^3$ that are a fixed distance from a fixed line. Find a parametrization of the right circular cylinder:

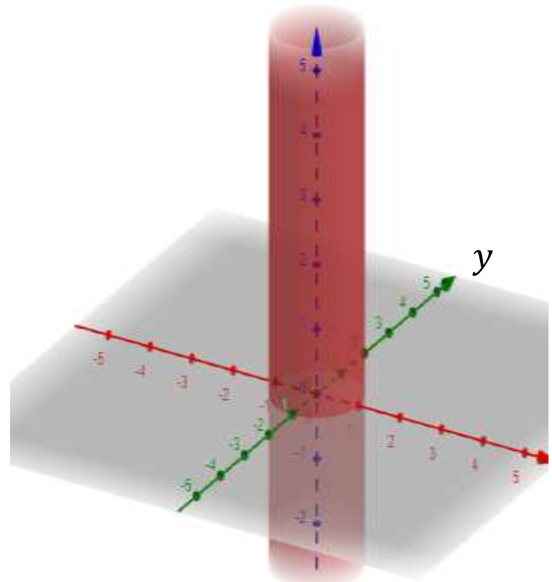
$$S = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 = 1\}.$$

One simple parametrization of S (there are an infinite number of parametrizations) is:

$$\vec{\Phi}(u, v) = (\cos u\,, \sin u\,, v);$$
$$0 \leq u \leq 2\pi, \quad v \in \mathbb{R}.$$

Notice if $x = \cos u\,, y = \sin u$, then

$x^2 + y^2 = \cos^2 u + \sin^2 u = 1.$



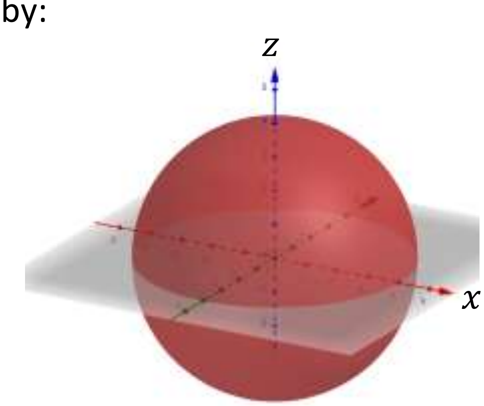
Ex. The sphere of radius R, centered at $(0,0,0)$:

$$S = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 + z^2 = R^2\}$$

can be parametrized (in spherical coordinates) by:

$$x = R\cos\theta\sin\phi$$
$$y = R\sin\theta\sin\phi$$
$$z = R\cos\phi$$

where $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$,



or equivalently:

$$\vec{\Phi}(\phi, \theta) = (R\cos\theta\sin\phi\,, R\sin\theta\sin\phi\,, R\cos\phi)$$

where $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$.

Notice:

$$x^2 + y^2 + z^2 = (R\cos\theta\sin\phi)^2 + (R\sin\theta\sin\phi)^2 + (R\cos\phi)^2$$
$$= R^2\sin^2\phi\,(\cos^2\theta + \sin^2\theta) + (R\cos\phi)^2$$
$$= R^2\sin^2\phi + R^2\cos^2\phi = R^2$$

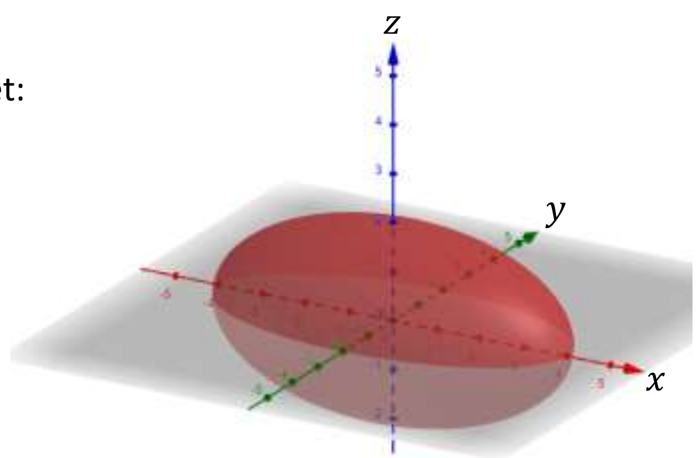
Ex. Find a parametrization of the ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Again, using spherical coordinates we get:

$$x = a(\cos\theta\sin\phi)$$
$$y = b\,(\sin\theta\sin\phi)$$
$$z = c(\cos\phi)$$

where $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$.



Equivalently we could write:

$$\vec{\Phi}(\phi,\theta) = (a\cos\theta\sin\phi\,, b\sin\theta\sin\phi\,, c\cos\phi)$$

where $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$.

Notice that:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = \cos^2\theta\sin^2\phi + \sin^2\theta\sin^2\phi + \cos^2\phi$$
$$= \sin^2\phi\,(\cos^2\theta + \sin^2\theta) + \cos^2\phi$$
$$= \sin^2\phi + \cos^2\phi = 1.$$

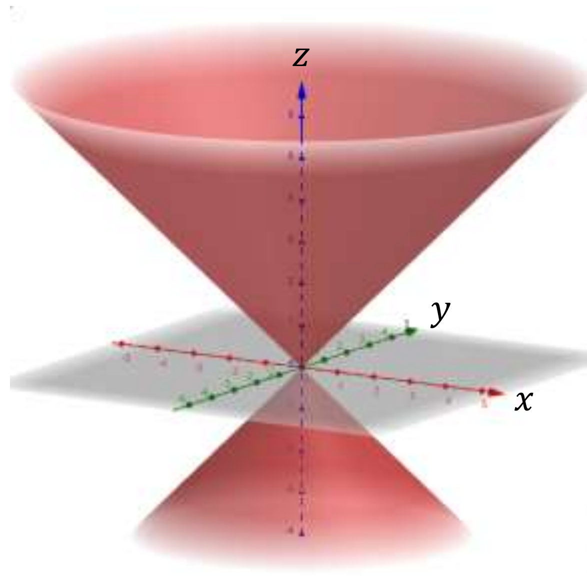
Ex. Find a parametrization of the circular cone $z^2 = x^2 + y^2$.

Let: $x = u\cos v\,;\quad y = u\sin v\,;\quad z = u$

where $u \in \mathbb{R}$ and $0 \le v \le 2\pi$.

Equivalently:

$$\vec{\Phi}(u,v) = (u\cos v\,,\ u\sin v\,,\ u);$$
$$u \in \mathbb{R},\quad 0 \le v \le 2\pi.$$



Notice that:

$$x^2 + y^2 = (u\cos v)^2 + (u\sin v)^2$$
$$= u^2\cos^2 v + u^2\sin^2 v$$
$$= u^2 = z^2.$$

If we had a general circular cone:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = \frac{z^2}{b^2}$$

we could parametrize this by:

$$\vec{\Phi}(u,v) = (au\cos v\,, au\sin v\,, bu);\quad u \in \mathbb{R},\quad 0 \le v \le 2\pi.$$

Note: a cone is actually not a surface because in a neighborhood of the point $(0,0,0)$ the set of points does not "look like" an open subset of $\mathbb{R}^2$. However, if you remove the point $(0,0,0)$ it is a surface.

Ex. Show that part of the hyperbolic paraboloid, $z = x^2 - y^2$, can be parametrized by:

$$\vec{\Phi}(u, v) = (v\cosh(u)\,, v\sinh(u)\,,\ v^2).$$

$$\begin{aligned} x^2 - y^2 &= (v\cosh(u))^2 - (v\sinh(u))^2 \\ &= v^2\cosh^2(u) - v^2\sinh^2(u) \\ &= v^2(\cosh^2(u) - \sinh^2(u)) \\ &= v^2 = z. \end{aligned}$$

Note that this parametrization is only good for part of the surface $z = x^2 - y^2$ where $z \geq 0$ as $z = v^2 \geq 0$.