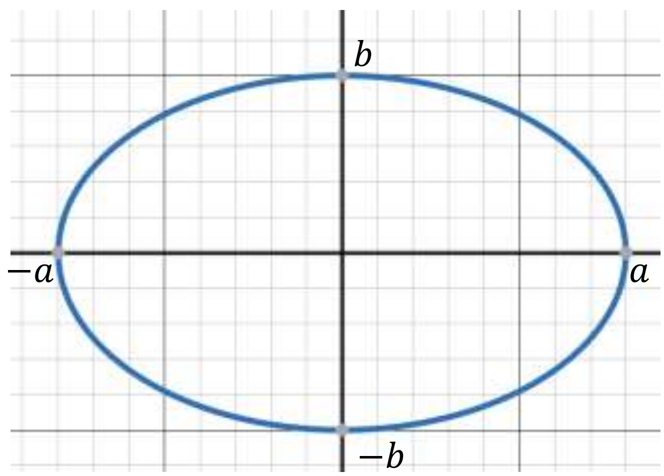
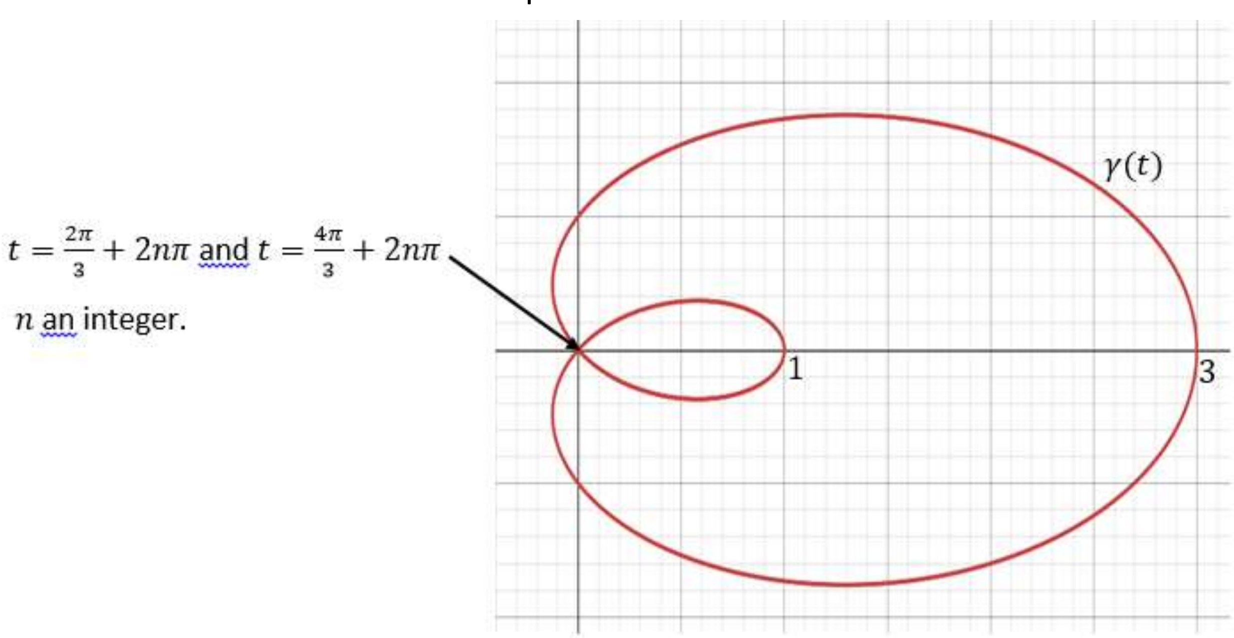
Simple Closed Curves in $\mathbb{R}^2$/The Isoperimetric Inequality/ The Four Vertex Theorem

Def. A **simple closed curve** in $\mathbb{R}^2$ is a closed curve in $\mathbb{R}^2$ that has no self intersections.

Ex. $\gamma(t) = (a\cos t\,,\ b\sin t), t \in \mathbb{R}$, an ellipse, is a simple closed curve in $\mathbb{R}^2$.

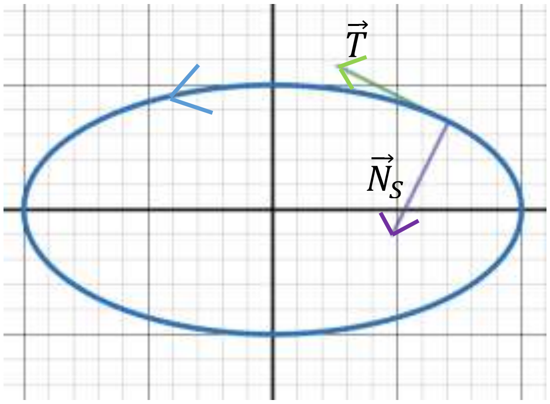


Ex. The limacon, $\gamma(t) = \big((1 + 2\cos t)\cos t), (1 + 2\cos t)\sin t\big), t \in \mathbb{R}$ is a closed curve but not a simple closed curve:

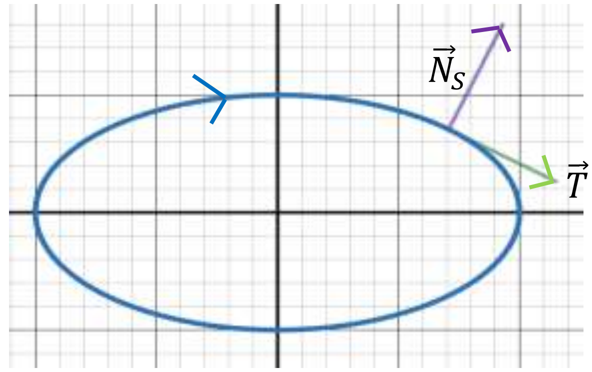


Every simple closed curve has a bounded interior and an unbounded exterior. For a simple closed curve, γ, we say γ is positively oriented if the signed normal $\overrightarrow{N_s}$ points into the interior of γ at every point of γ.

Positively Oriented



Negatively Oriented



Theorem: The total signed curvature of a simple closed curve in $\mathbb{R}^2$ is $\pm 2\pi$

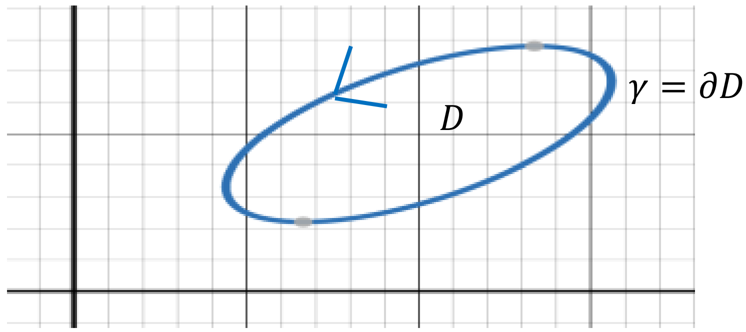
$$\int_0^l \kappa_s(s)\, ds = \pm 2\pi.$$

The Isoperimetric Inequality

Given a simple closed curve γ of fixed length $l = l(\gamma)$, what can we say about the area, $A(\gamma)$, enclosed by γ? The isoperimetric inequality gives us an upper bound on $A(\gamma)$. However, we will need a collorary of Green's theorem to establish this inequality.

Green's Theorem: Let $f(x,y)$ and $g(x,y)$ be smooth functions and let γ be a positively oriented simple closed curve that bounds a region $D \subseteq \mathbb{R}^2$. Then:

$$\iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\, dy = \int_\gamma f(x,y)dx + g(x,y)dy\ .$$



Cor: Let $\gamma(t) = \big(x(t), y(t)\big)$ be a positively oriented closed curve of length l bounding a region D, then:

$$\text{area}(D) = \int_\gamma x\, dy = -\int_\gamma y\, dx = \frac{1}{2}\int_\gamma (-y\, dx + x\, dy)\ .$$

Proof: $\text{area}(D) = \iint_D 1\, dxdy.$

Let $g(x,y) = x, \quad f(x,y) = 0$, then $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 1.$

Now apply Green's Theorem:

$$\text{area }(D) = \iint_D 1\, dxdy = \int_\gamma f(x,y)dx + g(x,y)dy = \int_\gamma x\, dy.$$

Now let $f(x,y) = -y, \quad g(x,y) = 0$, then $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 1.$

Now apply Green's Theorem:

$$\text{area}(D) = \iint_D 1\, dxdy = \int_\gamma f(x,y)dx + g(x,y)dy = -\int_\gamma y\, dx\ .$$

Finally, $\int_\gamma \; x\,dy - \int_\gamma \; y\,dx = 2(\text{area of } D)$ so we can write:

$$\frac{1}{2}\int_\gamma \; (x\,dy - y\,dx) = \text{area of } D.$$

Theorem (Isoperimetric Inequality): Let γ be a smooth simple closed curve in $\mathbb{R}^2$ with $l(\gamma) =$ length of γ and $A(\gamma) =$ area bounded by γ. Then:

$$A(\gamma) \leq \frac{1}{4\pi}\big(l(\gamma)\big)^2$$

and this equality holds if, and only if, γ is a circle.

Proof (We will prove this for regular curves, but it's actually true for C^1 curves):
Let L_1 and L_2 be parallel lines that are tangent to γ such that γ is contained between them.

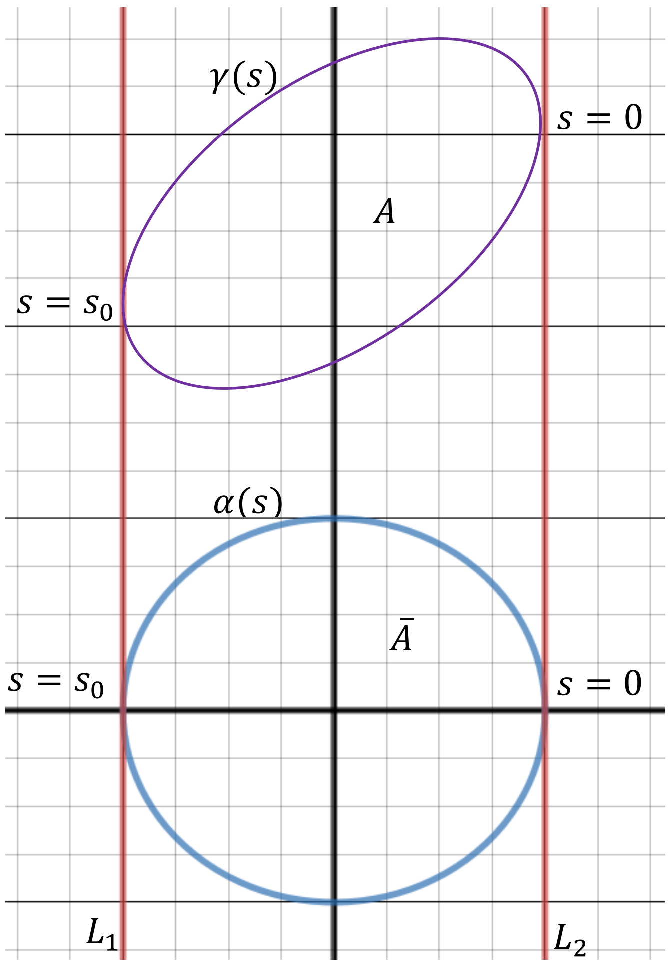
Let α be a circle that is tangent to L_1 and L_2 that doesn't intersect γ.

Let γ be parametrized by its arc length, s, and let $l = l(\gamma)$.

So $\gamma(s) = \big(x(s), y(s)\big)$, and we can parametrized the circle by:

$\alpha(s) = \big(\bar{x}(s), \bar{y}(s)\big) = \big(x(s), \bar{y}(s)\big)$ i.e. take $\bar{x}(s) = x(s)$.

So, $x^2 + \bar{y}^2 = r^2$.



$$\bar{x}(0) = x(0) = r$$

$$\bar{x}(s_0) = x(s_0) = -r$$

$$\bar{x}^2 + \bar{y}^2 = r^2$$

By Green's Theorem, the area bounded by γ, A, is:

$$A = \int_\gamma \; x\, dy = \int_0^l x \frac{dy}{ds}\, ds.$$

The area bounded by the circle α, $\bar{A}$, is:

$$\bar{A} = \pi r^2 = -\int_\alpha \; \bar{y}\, dx = -\int_0^l \bar{y} \frac{dx}{ds}\, ds.$$

Adding the two areas we get:

$$A + \pi r^2 = \int_0^l (x\, y' - x'\bar{y})\, ds \leq \int_0^l \sqrt{(xy' - x'\bar{y})^2}\; ds.$$

Claim: $(xy' - x'\bar{y})^2 \leq (x^2 + \bar{y}^2)((x')^2 + (y')^2)$.

$$(xy' - x'\bar{y})^2 = x^2(y')^2 - 2xy'x'\bar{y} + (x')^2\bar{y}^2.$$

But $0 \leq (\bar{y}y' + x\,x')^2 = \bar{y}^2(y')^2 + 2\bar{y}y'x'x + x^2(x')^2 \qquad (1)$

so we can write:

$$-2xy'x'\bar{y} \leq \bar{y}^2(y')^2 + x^2(x')^2.$$

$$(xy' - x'\bar{y})^2 \leq x^2(y')^2 + x^2(x')^2 + (\bar{y}^2)(y')^2 + (x')^2\bar{y}^2$$

$$= (x^2 + \bar{y}^2)((x')^2 + (y')^2).$$

So

$$(xy' - x'\bar{y})^2 \leq (x^2 + \bar{y}^2)((x')^2 + (y')^2).$$

Now we have:

$$\begin{aligned} A + \pi r^2 &\leq \int_0^l \sqrt{(xy' - x'\bar{y})^2}\, ds \\ &\leq \int_0^l \sqrt{(x^2 + \bar{y}^2)((x')^2 + (y')^2)}\, ds \\ &= \int_0^l \sqrt{x^2 + \bar{y}^2}\, ds \qquad \text{(since } \gamma \text{ is unit speed)} \\ &= \int_0^l \sqrt{\bar{x}^2 + \bar{y}^2}\, ds = lr \qquad \text{(since } \sqrt{\bar{x}^2 + \bar{y}^2} = r\text{)} \end{aligned}$$

Now dividing by 2 we get:

$$\frac{1}{2}(A + \pi r^2) \leq \frac{1}{2} lr\ .$$

But the geometric mean of two positive numbers is less than or equal to the arithmetic mean:

$$(*) \qquad \sqrt{(\pi r^2)(A)} \leq \frac{1}{2}(A + \pi r^2) \leq \frac{1}{2} lr$$

So, $\qquad (\pi r^2)A \leq \frac{1}{4} l^2 r^2 \quad$ and $\quad A \leq \frac{1}{4\pi} l^2.$

For the equality to hold, $(*)$ implies that $A = \pi r^2$ and $l = 2\pi r$.

From (1), $xx' + \bar{y}y' = 0$. But differentiating $x^2 + \bar{y}^2 = r^2$ gives:

$$xx' + \bar{y}\bar{y}' = 0 \Longrightarrow \ \bar{y}' = y'.$$

Thus, $y(s) = \bar{y}(s) +$ constant.

And since $x = \bar{x}$, and $\gamma = \big(x(s), y(s)\big)$,

we have:

$$r^2 = x^2 + \bar{y}^2 = x^2 + (y - const)^2,$$ and γ is a circle.

Ex. Does there exist a simple closed curve in $\mathbb{R}^2$ with length equal to 4 enclosing an area of 2?

No! By the isopermetric inequality:

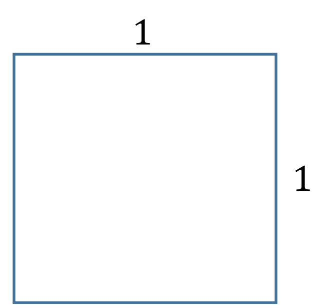
$$A \le \frac{1}{4\pi} l^2, \text{ but } \quad 2 > \frac{1}{4\pi}(4)^2 = \frac{4}{\pi}.$$

Ex. Does there exist a simple closed curve in $\mathbb{R}^2$ with length 4 enclosing an area of 1? If so, find an example.

$l = 4,\ A = 1,$ does satisfy the isopermetric inequality since

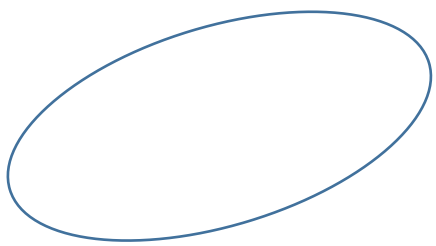
$$1 \le \frac{1}{4\pi}(4)^2 = \frac{4}{\pi}.$$

The isoperimetric inequality is actually true for piecewise smooth curves, so let's find a rectangle whose perimeter is 4 and whose area is 1. We could solve simutaneous equations to do that but clearly a square of side 1 works.

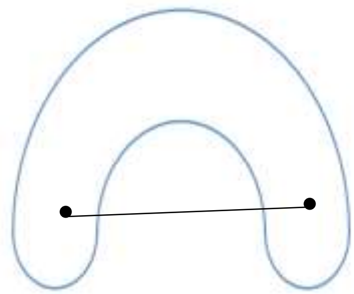


The Four Vertex Theorem

Def. A simple closed curve, γ, is called **convex** if a line segment joining any two points in the interior of γ lies entirely in the interior of γ.



Convex



Not Convex

Def. A vertex of a curve, $\gamma(t)$, in $\mathbb{R}^2$ is a point where $\frac{d\kappa_s}{dt} = 0$.

Recall that we saw earlier:

$$\kappa_s = \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{\frac{3}{2}}} \qquad \text{where } \gamma(t) = \big(x(t), y(t)\big).$$

Ex. Find all of the vertices of the ellipse $\gamma(t) = (3\cos t, 2\sin t)$.

$$x(t) = 3\cos t\,, \quad x' = -3\sin t, \quad x'' = -3\cos t$$

$$y(t) = 2\sin t\,, \quad y' = 2\cos t, \quad y'' = -2\sin t$$

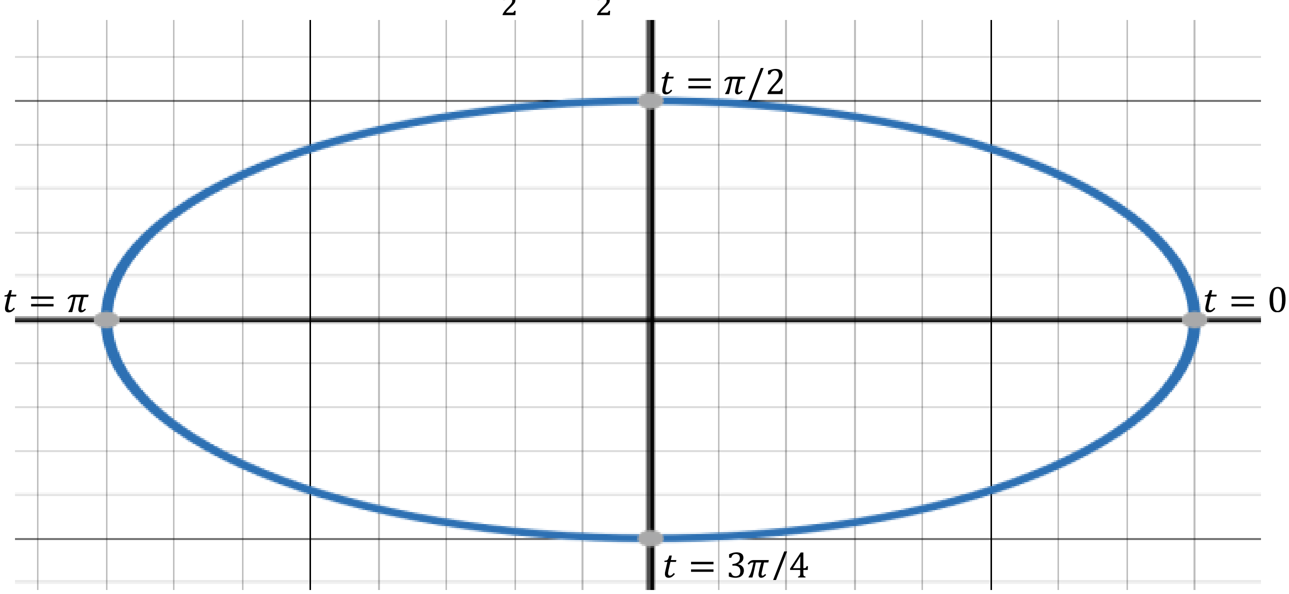
$$\kappa_s = \frac{(-3\sin t)(-2\sin t) - (-3\cos t)(2\cos t)}{((-3\sin t)^2 + (2\cos t)^2)^{\frac{3}{2}}}$$

$$\kappa_s = \frac{(3)(2)}{(3^2\sin^2 t + 2^2\cos^2 t)^{\frac{3}{2}}} = 6(9\sin^2 t + 4\cos^2 t)^{-\frac{3}{2}}$$

$$\frac{d\kappa_s}{dt} = -9(9\sin^2 t + 4\cos^2 t)^{-\frac{5}{2}}(18(\sin t)(\cos t) - 8(\sin t)(\cos t))$$

$$\frac{d\kappa_s}{dt} = \frac{-90\sin t\cos t}{(9\sin^2 t + 4\cos^2 t)^{\frac{5}{2}}} = 0\,;$$ where $\sin t = 0$ or $\cos t = 0$.

So verticies when $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.



Theorem (Four Vertex Theorem): Every convex simple closed curve in $\mathbb{R}^2$ has at least four vertices.

Ex. Find the vertices of $y = sinx$.

$$\gamma(t) = (t, sint)$$

$$x(t) = t \qquad y(t) = sint$$

$$x'(t) = 1 \qquad y'(t) = cost$$

$$x''(t) = 0 \qquad y''(t) = -sint.$$

$$\kappa_s = \frac{x'y''-x''y'}{((x')^2+(y')^2)^{\frac{3}{2}}} = \frac{(1)(-sint)}{(1+\cos^2 t)^{\frac{3}{2}}} = \frac{-sint}{(1+\cos^2 t)^{\frac{3}{2}}}$$

$$\frac{d\kappa_s}{dt} = \frac{(1+\cos^2 t)^{\frac{3}{2}}(-cost)-(-sint)(\frac{3}{2})(1+\cos^2 t)^{\frac{1}{2}}(2cost)(-sint)}{(1+\cos^2 t)^3}$$

$$= \frac{(1+\cos^2 t)^{\frac{1}{2}}[-(1+\cos^2 t)(cost)-3\sin^2 t(cost)]}{(1+\cos^2 t)^3}$$

$$= \frac{cost(-1-\cos^2 t-3\sin^2 t)}{(1+\cos^2 t)^{\frac{5}{2}}} = 0$$

$(-1 - \cos^2 t - 3\sin^2 t) < 0$ for all t.

$(1 + \cos^2 t)^{\frac{5}{2}} > 0$ for all t.

$cost = 0 \Rightarrow t = \frac{2n+1}{2}\pi; \quad n \in \mathbb{Z}$ are the vertices.

Ex. Find the vertices of $\gamma(t) = (e^t cost, e^t sint)$.

$$x(t) = e^t cost$$

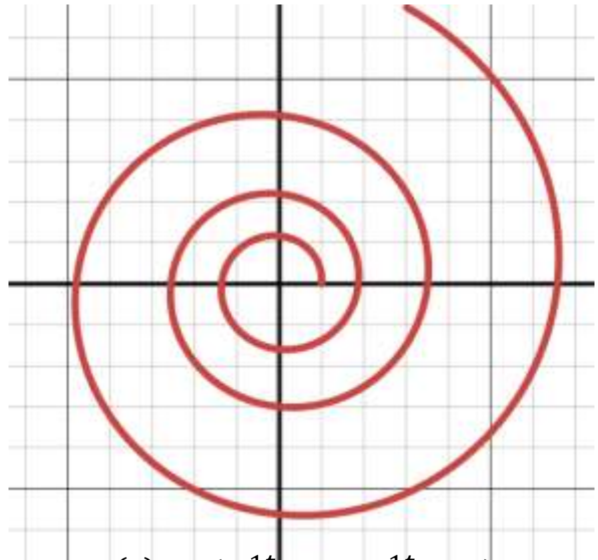
$$x'(t) = -e^t sint + e^t cost$$

$$x''(t) = -2e^t sint$$

$$y(t) = e^t sint$$

$$y'(t) = e^t cost + e^t sint$$

$$y''(t) = 2e^t cost.$$



$\gamma(t) = (e^{.1t} cost, e^{.1t} sint)$.

$$\kappa_s = \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{\frac{3}{2}}}$$

$$\kappa_s = \frac{\left(e^t(-sint+cost)\right)\left(2e^t cost\right) - (-2e^t sint)(e^t)(cost+sint)}{[e^{2t}(cost-sint)^2 + e^{2t}(cost+sint)^2]^{\frac{3}{2}}}$$

$$\kappa_s = \frac{2e^{2t}}{e^{3t}(2)^{\frac{3}{2}}} = \frac{1}{e^t\sqrt{2}} = \frac{\sqrt{2}}{2}e^{-t}$$

$\frac{d\kappa_s}{dt} = -\frac{\sqrt{2}}{2}e^{-t} \neq 0,$ so no vertices.

Notice that since the curvature of a circle is constant, every point of a circle is a vertex.

Ex. Find the vertices of the limacon $\gamma(t) = ((1 + 2cost)cost, (1 + 2cost)sint)$.

$$x(t) = (1 + 2cost)cost \qquad y(t) = (1 + 2cost)sint$$

$$x'(t) = -sint - 4sintcost \qquad y'(t) = cost + 2(\cos^2 t - \sin^2 t)$$

$$= -sint - 2sin2t \qquad = cost + 2cos2t$$

$$x''(t) = -cost - 4cos2t \qquad y''(t) = -sint - 4sin2t$$

$$\kappa_s = \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{\frac{3}{2}}}$$

$$= \frac{(-sint-2sin2t)(-sint-4sin2t)-(-cost-4cos2t)(cost+2cos2t)}{((-sint-2sin2t)^2+(cost+2cos2t)^2)^{\frac{3}{2}}}$$

$$= \frac{9 + 6(sint)sin2t + 6(cost)cos2t}{(5 + 4(sint)sin2t + 4(cost)cos2t)^{\frac{3}{2}}}$$

Notice that:

$$(sint)sin2t + (cost)cos2t = 2\sin^2 t(cost) + cost(1 - 2\sin^2 t)$$

$$= cost.$$

So we have: $\kappa_s = \frac{9+6cost}{(5+4cost)^{\frac{3}{2}}}$.

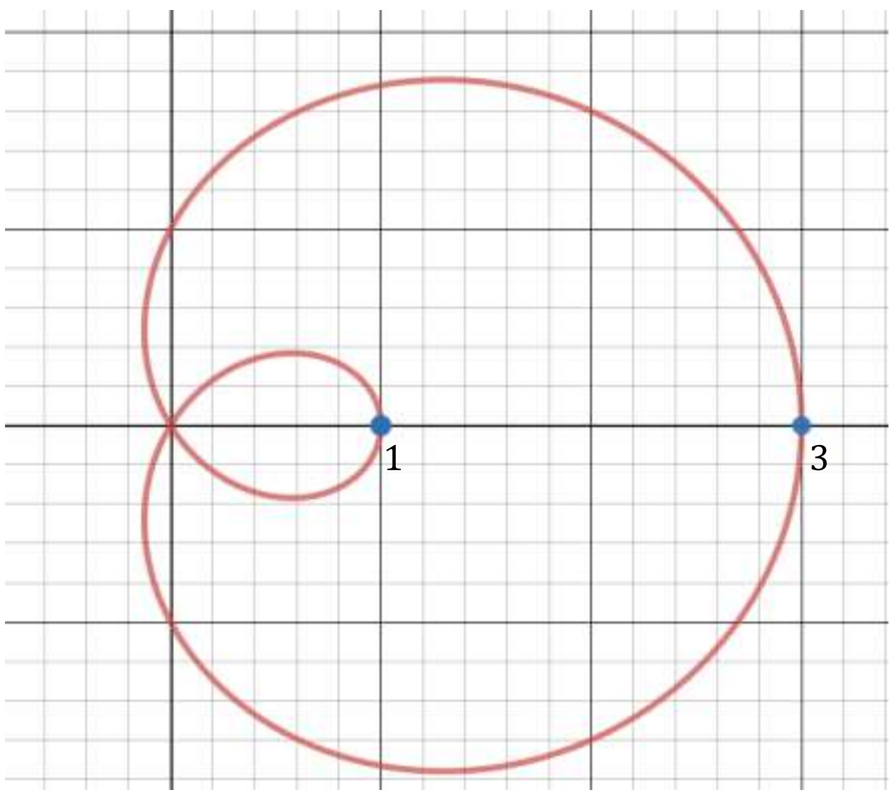
$$\frac{d\kappa_s}{dt} = \frac{24sint+12sintcost}{(5+4cost)^{\frac{3}{2}}} = 0$$

$$\Rightarrow \qquad 24sint + 12sintcost = 0$$

$$12sint(2 + cost) = 0.$$

$t = 0, \pi$ (since the curve is 2π periodic).

Thus this limacon has vertices at $\gamma(0) = (3,0)$, $\gamma(\pi) = (1,0)$.



Note: The limacon does not violate the 4 vertex theorem because is in not a simple closed curve.