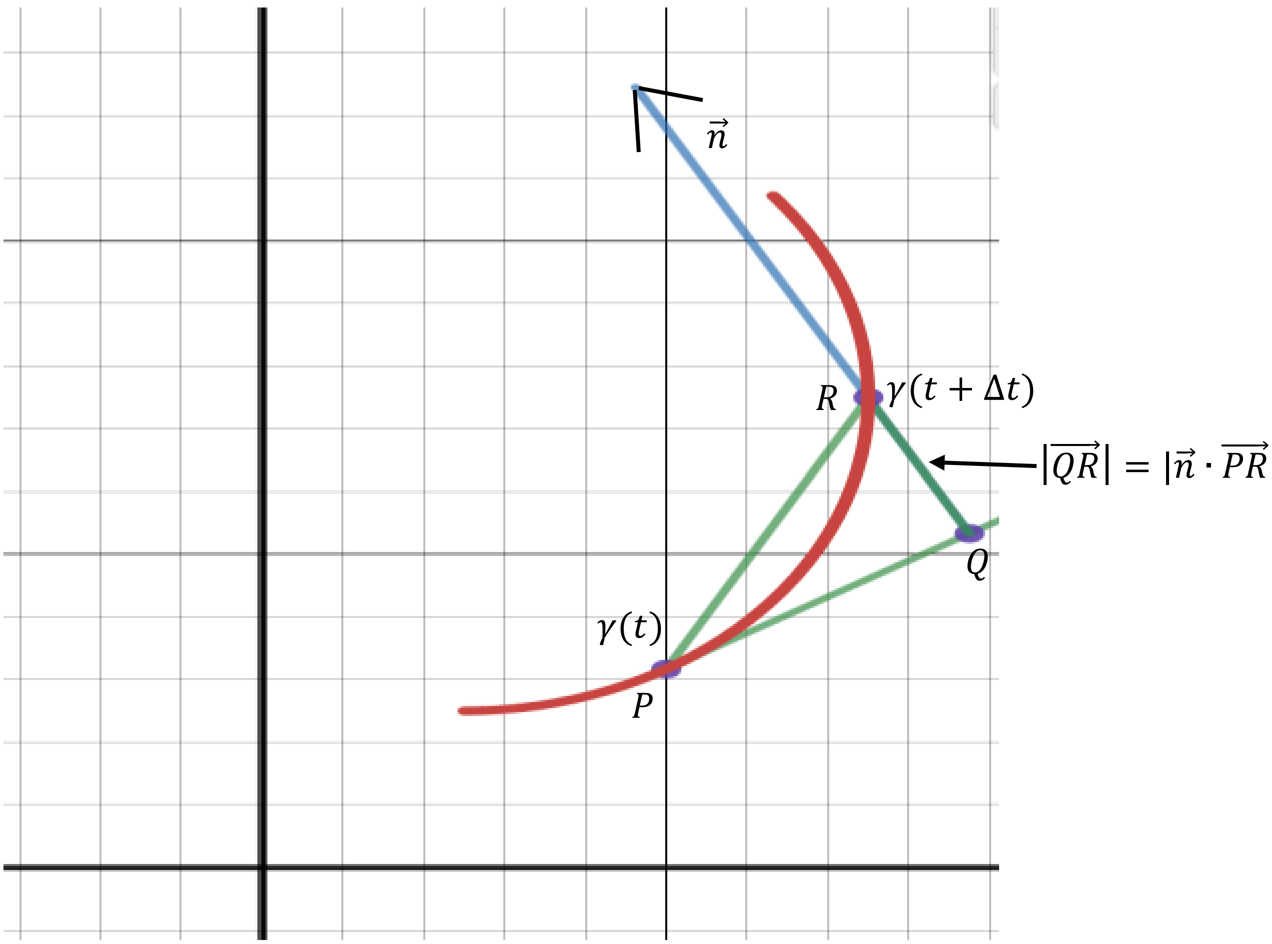
Curvature of Curves

Let's start with curves in $\mathbb{R}^2$. The curvature of a curve should measure the extent to which it is contained in a line (i.e. a line should have zero curvature).

Let $\gamma(t)$ be a unit speed curve in $\mathbb{R}^2$. Notice that as t goes to $t + \Delta t$ the amount the curve γ moves away from the tangent line at t is given by:

$$|(\gamma(t + \Delta t) - \gamma(t)) \cdot \vec{n}|$$

where $\vec{n}$ is a unit vector perpendicular to the tangent vector, $\gamma'(t)$, of $\gamma(t)$ at t.



By Taylor's Theorem:

$$\gamma(t+\Delta t) = \gamma(t) + \gamma'(t)(\Delta t) + \tfrac{1}{2}\big(\gamma''(t)\big)(\Delta t)^2 + \text{remainder}$$

where $\frac{(\text{remainder})}{(\Delta t)^2}$ goes to zero as Δt goes to zero.

$$\gamma(t+\Delta t) - \gamma(t) = \gamma'(t)(\Delta t) + \tfrac{1}{2}\big(\gamma''(t)\big)(\Delta t)^2 + \text{remainder}$$

$$\big(\gamma(t+\Delta t) - \gamma(t)\big)\cdot\vec{n} = (\gamma'(t)\cdot\vec{n})(\Delta t) + \tfrac{1}{2}(\gamma''(t)\cdot\vec{n})(\Delta t)^2 + \text{remainder}\cdot\vec{n}$$

$\gamma'(t)$ is tangent to $\gamma(t)$ at t so $\gamma'(t)\cdot\vec{n} = 0.$

Since $\gamma(t)$ is unit speed: $\gamma'\cdot\gamma' = 1.$

Differentiating this equation we get:

$$\gamma'\cdot\gamma'' + \gamma''\cdot\gamma' = 0$$

$$\gamma'\cdot\gamma'' = 0.$$

So $\gamma''(t)$ is also perpendicular to $\gamma'(t)$ so it's parallel to $\vec{n}$. Thus:

$$\gamma''(t) = \pm\|\gamma''(t)\|\vec{n}.$$

Thus we have:

$$(\gamma(t+\Delta t) - \gamma(t))\cdot\vec{n} = \pm\tfrac{1}{2}\|\gamma''(t)\|(\Delta t)^2 + \text{remainder}\cdot\vec{n}.$$

This suggests the following definition:

Def: If γ is a unit speed curve in $\mathbb{R}^3$ (or $\mathbb{R}^2$) with parameter t, then its **curvature, $\boldsymbol{\kappa(t)}$**, at $\gamma(t)$ is defined as $\|\gamma''(t)\|$.

Notice that if $\kappa(t) = \|\gamma''(t)\| = 0$ everywhere, then if $\gamma(t) = (a(t), b(t), d(t))$ we have:

$$a''(t) = 0 \Longrightarrow a(t) = c_1 t + c_2$$

$$b''(t) = 0 \Longrightarrow b(t) = c_3 t + c_4$$
$$d''(t) = 0 \Longrightarrow d(t) = c_5 t + c_6$$

So $\gamma(t) = (c_1 t + c_2,\ c_3 t + c_4,\ c_5 t + c_6\ )$, which is a line.

Ex. Find the curvature of a circle of radius R and center (a, b) in $\mathbb{R}^2$.

First notice that a unit speed parameterization of this circle is given by:

$$\gamma(t) = \left(a + R\cos\left(\frac{t}{R}\right), b + R\sin\left(\frac{t}{R}\right)\right)$$

$$\gamma'(t) = \left(-\sin\left(\frac{t}{R}\right), \cos\left(\frac{t}{R}\right)\right)$$

$$\|\gamma'(t)\| = \sqrt{\left(-\sin\left(\frac{t}{R}\right)\right)^2 + \left(\cos\left(\frac{t}{R}\right)\right)^2} = 1.$$

Now $\kappa(t) = \|\gamma''(t)\|$ so we can write:

$$\gamma''(t) = \left(-\frac{1}{R}\cos\left(\frac{t}{R}\right), -\frac{1}{R}\sin\left(\frac{t}{R}\right)\right)$$

$$\kappa(t) = \|\gamma''(t)\| = \sqrt{\left(-\frac{1}{R}\cos\left(\frac{t}{R}\right)\right)^2 + \left(-\frac{1}{R}\sin\left(\frac{t}{R}\right)\right)^2} = \frac{1}{R}.$$

This formula for curvature is fine but only applies to unit speed parameterizations, which may be very difficult (or impossible) to find. How do we calculate curvature for any smooth curve?

Prop: Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$. Then its curvature is:

$$\kappa = \frac{\|\gamma'' \times \gamma'\|}{\|\gamma'\|^3}$$

where $\times$ is the cross product of the vectors in $\mathbb{R}^3$.

Curves in $\mathbb{R}^2$ can be viewed as curves in $\mathbb{R}^3$ so this formula still applies to curves in $\mathbb{R}^2$.

Proof: Let s be a unit speed parameter for γ (and t is any parameter for γ). By the chain rule:

$$\gamma'(t) = \frac{d\gamma}{dt} = \frac{d\gamma}{ds}\frac{ds}{dt}$$

$$\Longrightarrow \qquad \frac{d\gamma}{ds} = \frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}}$$

$$\kappa = \left\|\frac{d^2\gamma}{ds^2}\right\| = \left\|\frac{d}{ds}\left(\frac{d\gamma}{ds}\right)\right\| = \left\|\frac{d}{ds}\left(\frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}}\right)\right\| = \left\|\frac{\left(\frac{d}{dt}\left(\frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}}\right)\right)}{\frac{ds}{dt}}\right\|$$

$$= \left\|\frac{\left(\frac{ds}{dt}\right)\left(\frac{d^2\gamma}{dt^2}\right) - \left(\frac{d\gamma}{dt}\right)\left(\frac{d^2s}{dt^2}\right)}{\left(\frac{ds}{dt}\right)^3}\right\|. \qquad (*)$$

Now notice that:

#1. $$\frac{d\gamma}{dt}\cdot\frac{d\gamma}{dt} = \left(\frac{ds}{dt}\frac{d\gamma}{ds}\right)\cdot\left(\frac{ds}{dt}\frac{d\gamma}{ds}\right) = \left(\frac{ds}{dt}\right)^2\left(\frac{d\gamma}{ds}\cdot\frac{d\gamma}{ds}\right) = \left(\frac{ds}{dt}\right)^2$$

Now differentiate both sides with respect to t:

#2. $$\frac{d^2\gamma}{dt^2}\cdot\frac{d\gamma}{dt} + \frac{d\gamma}{dt}\cdot\frac{d^2\gamma}{dt^2} = 2\left(\frac{ds}{dt}\right)\left(\frac{d^2 s}{dt^2}\right)$$

$$\frac{d\gamma}{dt}\cdot\frac{d^2\gamma}{dt^2} = \frac{ds}{dt}\left(\frac{d^2 s}{dt^2}\right)$$

So if we multiply $(*)$ by $\frac{ds}{dt}/\frac{ds}{dt}$ and use equations #1 and #2 we get:

$$\kappa = \left\| \frac{\left(\frac{ds}{dt}\right)^2\left(\frac{d^2\gamma}{dt^2}\right) - \left(\frac{d^2 s}{dt^2}\right)\left(\frac{ds}{dt}\right)\left(\frac{d\gamma}{dt}\right)}{\left(\frac{ds}{dt}\right)^4} \right\| = \left\| \frac{\left(\left(\frac{d\gamma}{dt}\right)\bullet\left(\frac{d\gamma}{dt}\right)\right)\left(\frac{d^2\gamma}{dt^2}\right) - \left(\frac{d\gamma}{dt}\bullet\frac{d^2\gamma}{dt^2}\right)\left(\frac{d\gamma}{dt}\right)}{\left\|\frac{d\gamma}{dt}\right\|^4} \right\|$$

Using the vector identity:

$$\vec{v}\times(\vec{w}\times\vec{u}) = (\vec{v}\cdot\vec{u})\vec{w} - (\vec{v}\cdot\vec{w})\vec{u}$$

We get:

$$\frac{d\gamma}{dt}\times\left(\frac{d^2\gamma}{dt^2}\times\frac{d\gamma}{dt}\right) = \left(\frac{d\gamma}{dt}\cdot\frac{d\gamma}{dt}\right)\frac{d^2\gamma}{dt^2} - \left(\frac{d\gamma}{dt}\cdot\frac{d^2\gamma}{dt^2}\right)\frac{d\gamma}{dt}.$$

$\frac{d\gamma}{dt}$ and $\frac{d^2\gamma}{dt^2} \times \frac{d\gamma}{dt}$ are perpendicular so:

$$\left\|\frac{d\gamma}{dt} \times \left(\frac{d^2\gamma}{dt^2} \times \frac{d\gamma}{dt}\right)\right\| = \left\|\frac{d\gamma}{dt}\right\| \left\|\frac{d^2\gamma}{dt^2} \times \frac{d\gamma}{dt}\right\|$$

since $\|A \times B\| = \|A\|\|B\| sin\theta.$

Thus we have:

$$\kappa = \frac{\left\|\frac{d\gamma}{dt} \times \left(\frac{d^2\gamma}{dt^2} \times \frac{d\gamma}{dt}\right)\right\|}{\left\|\frac{d\gamma}{dt}\right\|^4} = \frac{\left\|\frac{d^2\gamma}{dt^2} \times \frac{d\gamma}{dt}\right\|}{\left\|\frac{d\gamma}{dt}\right\|^3}.$$

Notice if $\gamma(t)$ is unit speed then $\frac{d\gamma}{dt}$ is perpendicular to $\frac{d^2\gamma}{dt^2}$ thus we have:

$$\left\|\frac{d^2\gamma}{dt^2} \times \frac{d\gamma}{dt}\right\| = \left\|\frac{d^2\gamma}{dt^2}\right\| \left\|\frac{d\gamma}{dt}\right\| = \left\|\frac{d^2\gamma}{dt^2}\right\|$$

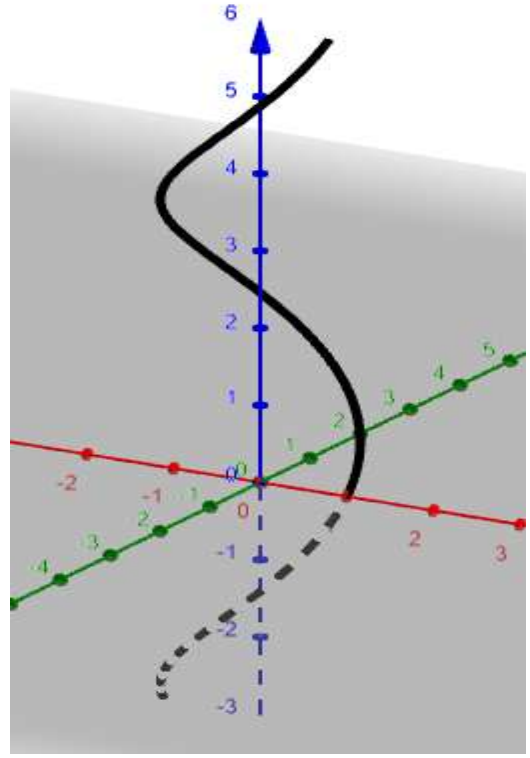
and

$$\kappa = \left\|\frac{d^2\gamma}{dt^2}\right\|.$$

The calculation of the curvature of a curve is independent of the regular parametrization used.

Ex. Find the curvature of the circular helix:

$$\gamma(t) = (a\cos t\,, a\sin t\,, bt)\;;\; t \in \mathbb{R}\;; \text{where } a, b \text{ are constants.}$$



$$\gamma'(t) = (-a\sin t\,, a\cos t\,, b)$$

$$\gamma''(t) = (-a\cos t\,, -a\sin t\,, 0)$$

$$\|\gamma'(t)\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + \ b^2)}$$

$$= \sqrt{a^2 + b^2}\,.$$

$$\gamma''(t)\ \times\ \gamma'(t) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -a\cos t & -a\sin t & 0 \\ -a\sin t & a\cos t & b \end{vmatrix}$$

$$= (-ab\sin t)\,\vec{\imath} - (-ab\cos t)\vec{\jmath} + (-a^2\cos^2 t - a^2\sin^2 t)\vec{k}$$

$$= (-ab\sin t)\,\vec{\imath} + (ab\cos t)\vec{\jmath} - a^2\vec{k}$$

$$\|\gamma''(t)\ \times\ \gamma'(t)\| = \sqrt{(-ab\sin t)^2 + (ab\cos t)^2 + (-a^2)^2}$$

$$= \sqrt{a^2b^2 + a^4} = |a|\sqrt{a^2 + b^2}$$

$$\kappa = \frac{\left\|\frac{d^2\gamma}{dt^2}\times\frac{d\gamma}{dt}\right\|}{\left\|\frac{d\gamma}{dt}\right\|^3} = \frac{|a|\sqrt{a^2+b^2}}{(a^2+b^2)^{\frac{3}{2}}} = \frac{|a|}{a^2+b^2}\;.$$

Thus, the circular helix has a constant curvature. Notice also that if $b = 0$,
$\kappa = \frac{|a|}{a^2} = \frac{1}{|a|}$, since the curve would then be a circle of radius $|a|$.

Ex. Find the curvature of $\gamma(t) = (t, \cosh(t))$.

Recall that:

$$\cosh(t) = \frac{e^t+e^{-t}}{2}, \quad \sinh(t) = \frac{e^t-e^{-t}}{2}$$

$$\cosh^2(t) - \sinh^2(t) = 1$$

$$\Rightarrow 1 + \sinh^2(t) = \cosh^2(t)$$

$$\frac{d}{dt}(\cosh(t)) = \sinh(t),$$

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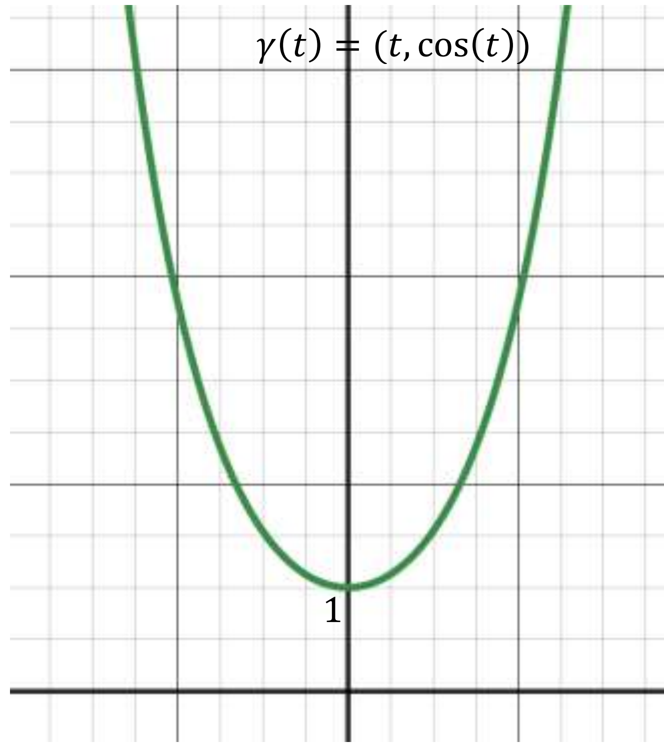

$$\gamma(t) = (t, \cosh(t), 0)$$

$$\gamma'(t) = (1, \sinh(t), 0)$$

$$\gamma''(t) = (0, \cosh(t), 0)$$

$$\|\gamma'(t)\| = \sqrt{1 + \sinh^2(t)} = \sqrt{\cosh^2(t)} = \cosh(t)$$

$$\gamma''(t) \times \gamma'(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & \cosh(t) & 0 \\ 1 & \sinh(t) & 0 \end{vmatrix} = (-\cosh(t))\,\vec{k}$$

$$\|\gamma''(t) \times \gamma'(t)\| = \sqrt{(-\cosh(t))^2} = \cosh(t)$$

$$\kappa = \frac{\left\|\frac{d^2\gamma}{dt^2} \times \frac{d\gamma}{dt}\right\|}{\left\|\frac{d\gamma}{dt}\right\|^3} = \frac{\cosh(t)}{\cosh^3(t)} = \frac{1}{\cosh^2(t)} = \operatorname{sech}^2(t).$$