Reparametrization of Curves and Closed Curves

As we saw earlier, there are many different parametrizations for the same curve.

Ex. $\gamma_1(t) = (\cos t, \sin t) \qquad 0 \le t \le 2\pi$

$\gamma_2(t) = (\cos 2t, \sin 2t) \quad 0 \le t \le \pi$

$\gamma_3(t) = (\sin t, \cos t) \qquad 0 \le t \le 2\pi$

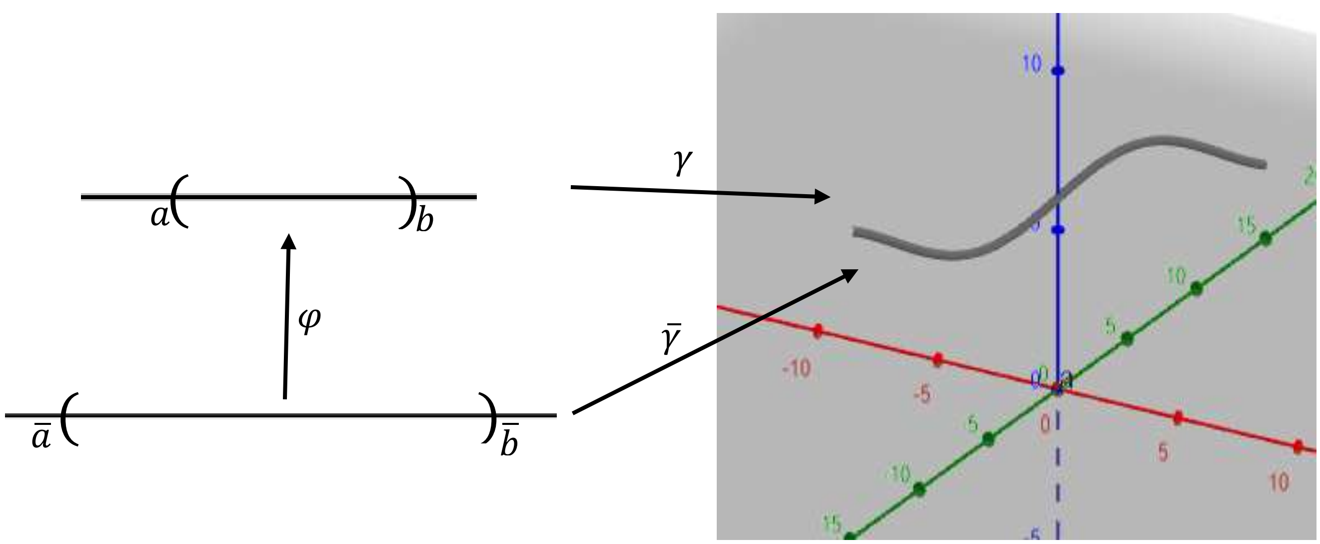
These are all representations of the circle $x^2 + y^2 = 1$.

Def. A parametrized curve $\bar{\gamma}: (\bar{a}, \bar{b}) \to \mathbb{R}^n$ is a reparametrization of a parametrized curve $\gamma: (a, b) \to \mathbb{R}^n$ if there is a smooth bijective map φ (i.e. 1-1 and onto)

$\varphi: (\bar{a}, \bar{b}) \to (a, b)$ such that the inverse map:

$\varphi^{-1}: (a, b) \to (\bar{a}, \bar{b})$ is also smooth and

$\bar{\gamma}(\bar{t}) = \gamma(\varphi(\bar{t}))$ for all $\bar{t} \in (\bar{a}, \bar{b})$.



Ex. $\gamma_2(t) = (\cos 2t\,, \sin 2t)$ and $\gamma_3(t) = (\sin t\,, \cos t)$ are both reparametrizations of $\gamma_1(t) = (\cos t\,, \sin t)$ since:

if $\varphi(t) = 2t$ then

$$\begin{aligned}\gamma_2(t) &= \gamma_1(\varphi(t)) \\ &= \gamma_1(2t) \\ &= (\cos 2t\,, \sin 2t).\end{aligned}$$

And if $\varphi(t) = \frac{\pi}{2} - t$ then

$$\begin{aligned}\gamma_3(t) = \gamma_1(\varphi(t)) &= \gamma_1\left(\frac{\pi}{2} - t\right) \\ &= (\cos\left(\frac{\pi}{2} - t\right), \sin(\frac{\pi}{2} - t)) \\ &= (\sin t, \cos t).\end{aligned}$$

It turns out that the analysis of a curve is simplified if it is parametrized in a way that has unit speed (i.e. $\|\gamma'(t)\| = 1$).

Def: A point $\gamma(t)$ of a parametrized curve γ is called a **regular point** if $\gamma'(t) \neq \vec{0}$. Otherwise, $\gamma'(t)$ is a **singular point** of γ . A curve is **regular** if all of its points are regular.

Ex. $\gamma_1(t) = (t, t^2)$ is a regular parametrization of $y = x^2$ because

$\gamma_1'(t) = (1, 2t) \neq (0{,}0)$ for any t.

However, $\gamma_2(t) = (t^3, t^6)$ is not a regular parametrization of

$y = x^2$ because $\gamma_2'(t) = (3t^2, 6t^5)$ which is $(0{,}0)$ at $t = 0$.

Prop: Any reparametrization of a regular curve is regular.

Proof: Let $\gamma(t)$ be a regular curve. We must show given a reparametrization, $\gamma(\varphi(t))$, that $\frac{d}{dt}(\gamma(\varphi(t)))$ exists and is never $\vec{0}$.

Let $\psi = \varphi^{-1}$ then $\psi(\varphi(t)) = t$. Using the chain rule we get:

$$\psi'(\varphi(t))\varphi'(t) = 1; \quad \text{thus } \varphi'(t) \neq 0.$$

Applying the chain rule to $\gamma(\varphi(t))$ we get:

$$\frac{d}{dt}\Big(\gamma(\varphi(t))\Big) = \gamma'(\varphi(t))\varphi'(t).$$

γ is regular so $\gamma' \neq \vec{0}$, and $\varphi'(t) \neq 0$, so $\frac{d}{dt}\Big(\gamma(\varphi(t))\Big) \neq \vec{0}$.

$\frac{d}{dt}(\gamma(\varphi(t)))$ also exists by the chain rule, so $\gamma(\varphi(t))$ is regular.

Prop: If $\gamma(t)$ is a regular curve then its arc length, s, starting at any point of γ, is a smooth function of t.

Proof: We saw earlier that $s = \int_{t_0}^{t} \|\gamma'(u)\|\, du$, so $\frac{ds}{dt} = \|\gamma'(t)\|$.

If $\gamma(t) = (u(t), v(t))$ and $\gamma'(t) = (u'(t), v'(t))$, then:

$$\frac{ds}{dt} = \sqrt{(u')^2 + (v')^2} = \left((u')^2 + (v')^2\right)^{\frac{1}{2}}$$

$$\frac{d^2s}{dt^2} = \frac{1}{2}\left((u')^2 + (v')^2\right)^{-\frac{1}{2}}\left(2(u')u'' + 2(v')v''\right).$$

Now if $f(x) = \sqrt{x}$ for $x > 0$, then we know $f^{(n)}(x)$ exists for all $x > 0, n \geq 1$.

Thus, since u, v are smooth and $(u')^2 + (v')^2 > 0$ (since $\gamma(t)$ is regular), $\frac{ds}{dt}$ is smooth for all t and hence so is $s(t)$.

Prop: A parametrized curve has a unit speed reparametrization if and only if it is regular.

Generally, we will be using the statement that if γ is regular then it has a unit speed reprametrization. This follows from the inverse function theorem.

Corallary: Let γ be a regular curve and let $\bar{\gamma}$ be a unit speed reparametrization of γ: $\bar{\gamma}\big(u(t)\big) = \gamma(t)$, for all t where u is a smooth function of t.

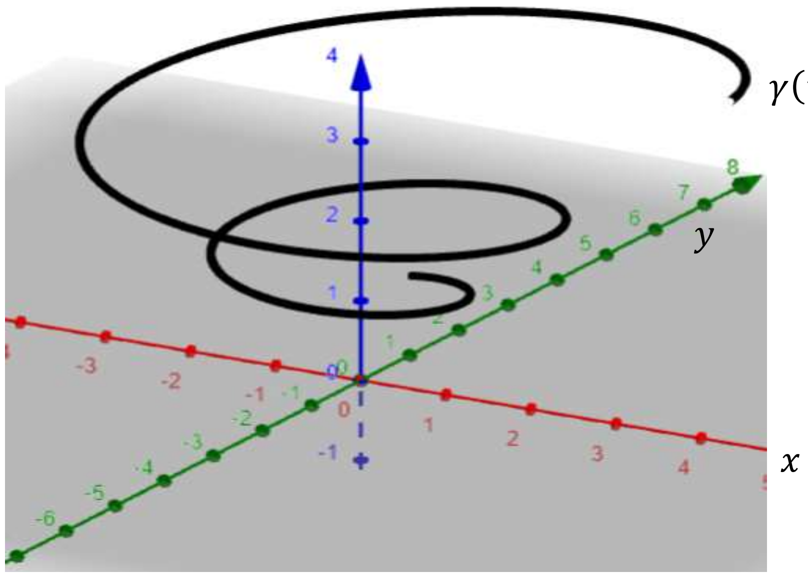
Then, if s is the arc length of γ (starting at any point), we have:

$u = \pm s + c;$ where c is a constant.

Conversely, if u is given by $u = \pm s + c$, then $\bar{\gamma}$ is a unit speed reparametrization of γ.

Thus we can think of any unit speed parametrization of a curve as being parametrized by (a translation of) arc length.

Ex. Show that $\gamma(t) = (e^t \sin t, e^t \cos t, e^t)$ is a regular curve for $t \in \mathbb{R}$ and find a unit speed reparametrization of $\gamma(t)$.



$\gamma(t) = (e^{.1t} \sin t, e^{.1t} \cos t, e^{.1t})$;

$t > 0$

$$\gamma'(t) = (e^t(\cos t + \sin t),\ \ e^t(\cos t - \sin t),\ \ e^t)$$

$\|\gamma'(t)\| = \sqrt{3}e^t \neq 0$ for any $t \in \mathbb{R}$

Thus $\gamma(t)$ is a regular curve for $t \in \mathbb{R}$.

The arc length starting at $\gamma(0) = (0,1,1)$ is written as:

$$s = \int_0^t \sqrt{3}e^u \ du = \sqrt{3}(e^t - 1)$$

To write $\gamma(t)$ in terms of s we need to solve $s = \sqrt{3}(e^t - 1)$ for t in terms of s.

To do this we must find the inverse function of

$$s(t) = \sqrt{3}(e^t - 1):$$

$$\frac{s}{\sqrt{3}} = e^t - 1$$

$$\frac{s}{\sqrt{3}} + 1 = e^t$$

$$\ln(\frac{s}{\sqrt{3}} + 1) = t.$$

Now to find $\gamma(t) = (e^t \sin t\,, e^t \cos t, e^t)$ in terms of s, substitute $t = \ln(\frac{s}{\sqrt{3}} + 1)$ into the formula for $\gamma(t)$.

Notice:

$$e^t = e^{\ln(\frac{s}{\sqrt{3}}+1)} = \frac{s}{\sqrt{3}} + 1.$$

Thus, $\gamma(t) = (e^t \sin t\,, e^t \cos t, e^t)$ becomes:

$$\bar{\gamma}(s) = \left(\left(\frac{s}{\sqrt{3}} + 1\right)\sin(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right), \left(\frac{s}{\sqrt{3}} + 1\right)\cos(\ln\left(\frac{s}{\sqrt{3}} + 1\right))), \frac{s}{\sqrt{3}} + 1) \ .$$

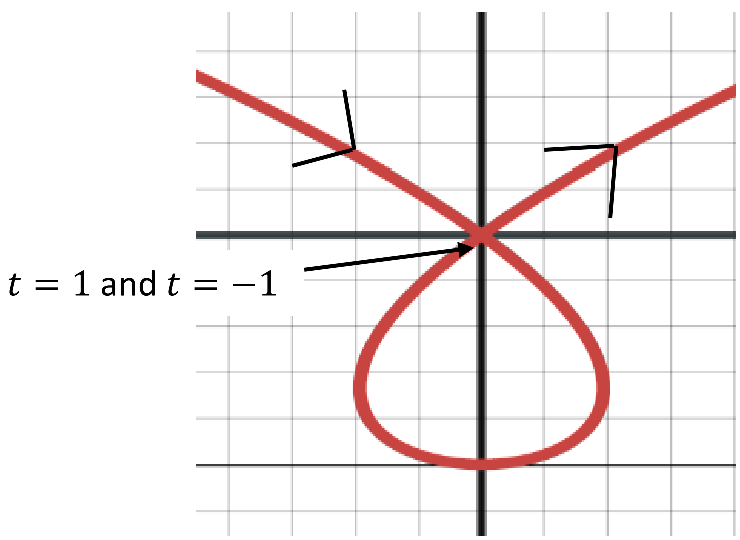
Note: In general it can be very difficult (or even impossible) to write a given regular curve $\gamma(t)$ in terms of elementary functions of a unit speed parameter s. This is because $s = \int_{t_0}^{t} \|\gamma'(u)\|\, du$ often can't be written in terms of elementary functions, and if it can, finding the inverse function of $s(t)$ could be very difficult.

Ex. Notice that the circle $x^2 + y^2 = 1$ given by $\gamma(t) = (\cos t, \sin t)$, where $t \in \mathbb{R}$, is already a unit speed parametrization.

$$\gamma'(t) = (-\sin t, \cos t)$$

$$\|\gamma'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1.$$

It's clear that some curves, like circles and ellipses, have the property that a point will return to its starting point and then retrace the same curve. However, one can have a curve where a point returns to its starting point but does not retrace the same curve. For example: $\gamma(t) = (t^3 - t, t^2 - 1)$; $t \in \mathbb{R}$.



Def. Let $\gamma: \mathbb{R} \to \mathbb{R}^n$ be a smooth curve and let $\mathrm{T} \in \mathbb{R}$.

We say γ is **T-periodic** if: $\gamma(t + \mathrm{T}) = \gamma(t)$ for all $t \in \mathbb{R}$.

If γ is not constant and is T-periodic for some $\mathrm{T} \neq 0$, then γ is said to be **closed.**

Notice that a T-periodic curve γ is determined by its restriction to any interval of length $|\mathrm{T}|$. So, closed curves can be thought of as:

$$\gamma: [a, b] \to \mathbb{R}^n$$

where $b - a = |\mathrm{T}|$ and $\gamma(b) = \gamma(a)$, as well as $\gamma^n(b) = \gamma^n(a)$.

Def. The **period** of a closed curve γ is the smallest positive number T such that γ is T-periodic.

Ex. $\gamma(t) = (\cos 4t, \sin 4t)$ is a closed curve (a circle of period $\frac{\pi}{2}$).

Def. A curve (closed or not) is said to have a **self-intersection at a point, $\boldsymbol{p}$**, on the curve if there exist parameter values $a \neq b$ such that:

i) $\gamma(a) = \gamma(b) = p$.
ii) If γ is closed with period T, then $a - b$ is not an integer multiple of T.

Ex. $\gamma(t) = (t^3 - t, t^2 - 1)$; $\quad t \in \mathbb{R}$ has a self intersection point at $t = \pm 1$, but is not a closed curve since there is no T such that $\gamma(t + \mathrm{T}) = \gamma(t)$ for all $t \in \mathbb{R}$.

Ex. The limacon given by:

$$\gamma(t) = ((1 + 2\cos t)\cos t\,, (1 + 2\cos t)\sin t) \;;\quad t \in \mathbb{R}$$

is a closed curve of period 2π and has a self-intersection point at $(0,0)$.

$t = \frac{2\pi}{3} + 2n\pi$ and $t = \frac{4\pi}{3} + 2n\pi$

n an integer.