The Gauss-Bonnet Theorem- HW Problems

1. In an earlier HW exercise we saw that the Gauss curvature function of the torus:

   $$\vec{\Phi}(u,v) = ((2+\cos(v))\cos(u)\,,(2+\cos(v))\sin(u)\,,\sin(v));$$

   where $(u,v) \in [0,2\pi) \times [0,2\pi)$

   is given by $\mathrm{K} = \frac{\cos(v)}{2+\cos(v)}$. The Euler characteristic of the torus, $\chi(T)$, is 0. Show that the Gauss-Bonnet theorem holds for this surface, i.e., show: $\iint_T \mathrm{KdS} = 2\pi\chi(T) = 0$.

   (recall $\iint_T \mathrm{K}dS = \int_{v=0}^{2\pi}\int_{u=0}^{2\pi} \mathrm{K}\left\|\vec{\Phi}_u \times \vec{\Phi}_v\right\| dudv$).

2. Consider the ellipsoid S given by

   $$\vec{\Phi}(u,v) = ((\cos(v))\sin(u)\,,(\sin(v)\sin(u)\,,2\cos(u));$$

   where $(u,v) \in [0,2\pi) \times [0,2\pi)$.

   (ie, the surface $x^2 + y^2 + \frac{z^2}{4} = 1$).

   It can be shown through direct calculation that the Gaussian curvature is: $\mathrm{K} = \frac{4}{(\cos^2 u + 4\sin^2 u)^2}$.

   An ellipsoid is homeomorphic to a sphere thus $\chi(S) = 2$.

   Show that the Gauss-Bonnet theorem holds for this surface. That is show: $\iint_S \mathrm{KdS} = 2\pi\chi(S) = 4\pi$.