The Gauss Equations and the Codazzi-Mainardi Equations

We have used the first fundamental form to measure the length of a curve and to find surface area. Then we used the first and second fundamental forms to measure different types of curvature of a surface (Gaussian, mean, normal, etc). It is natural to ask what the relationship is between components of the first fundamental form (E, F, and G) and components of the second fundamental form (L, M, and N). We will find these relationships in the Gauss Equations and the Codazzi-Mainardi equations.

Proposition (Gauss Equations 1):

Let $\vec{\Phi}(u, v)$ be a surface patch of a surface, $S \subseteq \mathbb{R}^3$, with first and second fundamental forms:

$$Edu^2 + 2Fdudv + Gdv^2 \text{ and } Ldu^2 + 2Mdudv + Ndv^2$$

Then:

$$\vec{\Phi}_{uu} = \Gamma_{11}^1 \vec{\Phi}_u + \Gamma_{11}^2 \vec{\Phi}_v + L\vec{N}$$

$$\vec{\Phi}_{uv} = \Gamma_{12}^1 \vec{\Phi}_u + \Gamma_{12}^2 \vec{\Phi}_v + M\vec{N}$$

$$\vec{\Phi}_{vv} = \Gamma_{22}^1 \vec{\Phi}_u + \Gamma_{22}^2 \vec{\Phi}_v + N\vec{N}$$

where:

$$\Gamma_{11}^1 = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)} \qquad \Gamma_{11}^2 = \frac{2FF_u - EE_v - FE_u}{2(EG - F^2)}$$

$$\Gamma_{21}^1 = \Gamma_{12}^1 = \frac{GE_v - FG_u}{2(EG - F^2)} \qquad \Gamma_{21}^2 = \Gamma_{12}^2 = \frac{EG_u - FE_v}{2(EG - F^2)}$$

$$\Gamma_{22}^1 = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)} \qquad \Gamma_{22}^2 = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}.$$

The six coefficients of $\vec{\Phi}_u$ and $\vec{\Phi}_v$ in the formulas for $\vec{\Phi}_{uu}, \vec{\Phi}_{uv}$, and $\vec{\Phi}_{vv}$ are called Christoffel symbols (i.e., the $\Gamma^{\mathrm{i}}_{\mathrm{jk}}s$). Notice that the formulas for the Christoffel symbols, $\Gamma^{\mathrm{i}}_{\mathrm{jk}}$, depend only on the components of the first fundamental form and their partial derivatives.

Proof: Since $\vec{\Phi}_u, \vec{\Phi}_v$ and $\vec{N}$ are a basis for $\mathbb{R}^3$, we can write $\vec{\Phi}_{uu}, \vec{\Phi}_{uv}$, and $\vec{\Phi}_{vv}$ in terms of $\vec{\Phi}_u, \vec{\Phi}_v$ and $\vec{N}$.

$$\begin{aligned} \vec{\Phi}_{uu} &= a_1\vec{\Phi}_u + b_1\vec{\Phi}_v + c_1\vec{N} \\ \vec{\Phi}_{uv} &= a_2\vec{\Phi}_u + b_2\vec{\Phi}_v + c_2\vec{N} \qquad (*) \\ \vec{\Phi}_{vv} &= a_3\vec{\Phi}_u + b_3\vec{\Phi}_v + c_3\vec{N} \end{aligned}$$

Taking the dot product of each equation with $\vec{N}$, we get:

$$\begin{aligned} L &= \vec{\Phi}_{uu} \cdot \vec{N} = \left(a_1\vec{\Phi}_u + b_1\vec{\Phi}_v + c_1\vec{N}\right) \cdot \vec{N} = c_1 \\ M &= \vec{\Phi}_{uv} \cdot \vec{N} = (a_2\vec{\Phi}_u + b_2\vec{\Phi}_v + c_2\vec{N}) \cdot \vec{N} = c_2 \\ N &= \vec{\Phi}_{vv} \cdot \vec{N} = \left(a_3\vec{\Phi}_u + b_3\vec{\Phi}_v + c_3\vec{N}\right) \cdot \vec{N} = c_3 \end{aligned}$$

If we take the three equations $(*)$ and dot them with $\vec{\Phi}_u$ and $\vec{\Phi}_v$ we get six equations in the six unknowns: $a_1, a_2, a_3, b_1, b_2, b_3$.

For example:

$$\begin{aligned} \vec{\Phi}_{uu} \cdot \vec{\Phi}_u &= \left(a_1\vec{\Phi}_u + b_1\vec{\Phi}_v + c_1\vec{N}\right) \cdot \vec{\Phi}_u \\ &= a_1\left(\vec{\Phi}_u \cdot \vec{\Phi}_u\right) + b_1\left(\vec{\Phi}_v \cdot \vec{\Phi}_u\right) = a_1E + b_1F. \end{aligned}$$

But, $\frac{\partial}{\partial u}(E) = \frac{\partial}{\partial u}\left(\vec{\Phi}_u \cdot \vec{\Phi}_u\right) = 2\vec{\Phi}_u \cdot \vec{\Phi}_{uu}$ or $\vec{\Phi}_u \cdot \vec{\Phi}_{uu} = \frac{1}{2}E_u$

So, $$\frac{1}{2}E_u = a_1E + b_1F.$$

Solving these six simultaneous equations for $a_1, a_2, a_3, b_1, b_2, b_3$ gives:

$$a_1 = \Gamma_{11}^1 \qquad b_1 = \Gamma_{11}^2$$
$$a_2 = \Gamma_{12}^1 \qquad b_2 = \Gamma_{12}^2$$
$$a_3 = \Gamma_{22}^1 \qquad b_3 = \Gamma_{22}^2.$$

Ex. Let $\vec{\Phi}(u,v) = (R\cos v \sin u, R\sin v \sin u\,, R\cos u)$ be a parametrization of the sphere $x^2 + y^2 + z^2 = R^2$. Find the 8 Christoffel symbols of $\vec{\Phi}$ and write $\vec{\Phi}_{uu}, \vec{\Phi}_{uv}$, and $\vec{\Phi}_{vv}$ in terms of $\vec{\Phi}_u, \vec{\Phi}_v$, and $\vec{N}$.

To calculate the Christoffel symbols we just need the components of the first fundamental form of $\vec{\Phi}$ (and their derivatives). To calculate the components of $\vec{\Phi}_{uu}, \vec{\Phi}_{uv}$, and $\vec{\Phi}_{vv}$ in terms of $\vec{\Phi}_u, \vec{\Phi}_v$, and $\vec{N}$ we need the components of the second fundamental form.

Recall that for the sphere of radius R, the first fundamental form is:

$$\begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2 u \end{pmatrix}; \quad E = R^2, \qquad F = 0, \qquad G = R^2\sin^2 u.$$

The second fundamental form:

$$\begin{pmatrix} -R & 0 \\ 0 & -R\sin^2 u \end{pmatrix}; \quad L = -R, \qquad M = 0, \qquad N = -R\sin^2 u.$$

Using the formulas for $\Gamma^{\mathrm{i}}_{\mathrm{jk}}$in the Gauss equations we get:

$$\Gamma^1_{11} = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)} = 0$$

$$\Gamma^2_{11} = \frac{2FF_u - EE_v - FE_u}{2(EG - F^2)} = 0$$

$$\Gamma^1_{21} = \Gamma^1_{12} = \frac{GE_v - FG_u}{2(EG - F^2)} = 0$$

$$\Gamma^2_{21} = \Gamma^2_{12} = \frac{EG_u - FE_v}{2(EG - F^2)} = \frac{R^2(2R^2 \sin u \cos u)}{2R^4 \sin^2 u} = \cot u$$

$$\Gamma^1_{22} = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)} = \frac{-R^2 \sin^2 u(2R^2 \sin u \cos u)}{2R^4 \sin^2 u} = -\sin u\,(\cos u)$$

$$\Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)} = 0.$$

So, now we can write:

$$\vec{\Phi}_{uu} = \Gamma^1_{11}\vec{\Phi}_u + \Gamma^2_{11}\vec{\Phi}_v + L\vec{N} = -R\vec{N}$$

$$\vec{\Phi}_{uv} = \Gamma^1_{12}\vec{\Phi}_u + \Gamma^2_{12}\vec{\Phi}_v + M\vec{N} = (\cot u)\vec{\Phi}_v$$

$$\vec{\Phi}_{vv} = \Gamma^1_{22}\vec{\Phi}_u + \Gamma^2_{22}\vec{\Phi}_v + N\vec{N} = -(\sin u)(\cos u)\vec{\Phi}_u - (R\sin^2 u)\vec{N}.$$

Proposition (Codazzi-Mainardi Equations):

Let $Edu^2 + 2Fdudv + Gdv^2$ and $Ldu^2 + 2Mdudv + Ndv^2$ be the first and second fundamental forms of a surface patch, $\overrightarrow{\Phi}(u, v)$. Then:

$$L_v - M_u = L(\Gamma_{12}^1) + M(\Gamma_{12}^2 - \Gamma_{11}^1) - N(\Gamma_{11}^2)$$

$$M_v - N_u = L(\Gamma_{22}^1) + M(\Gamma_{22}^2 - \Gamma_{12}^1) - N(\Gamma_{12}^2).$$

Proposition (Gauss Equations 2):

If K is the Gaussian curvature of a surface patch, $\overrightarrow{\Phi}(u, v)$, then:

$$E\mathrm{K} = (\Gamma_{11}^2)_v - (\Gamma_{12}^2)_u + \Gamma_{11}^1\Gamma_{12}^2 + \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{12}^1\Gamma_{11}^2 - (\Gamma_{12}^2)^2$$

$$F\mathrm{K} = (\Gamma_{12}^1)_u - (\Gamma_{11}^1)_v + \Gamma_{12}^2\Gamma_{12}^1 - \Gamma_{11}^2\Gamma_{22}^1$$

$$F\mathrm{K} = (\Gamma_{12}^2)_v - (\Gamma_{22}^2)_u + \Gamma_{12}^1\Gamma_{12}^2 - \Gamma_{22}^1\Gamma_{11}^2$$

$$G\mathrm{K} = (\Gamma_{22}^1)_u - (\Gamma_{12}^1)_v + \Gamma_{22}^1\Gamma_{11}^1 + \Gamma_{22}^2\Gamma_{12}^1 - (\Gamma_{12}^1)^2 - \Gamma_{12}^2\Gamma_{22}^1.$$

Proof of Codazzi-Mainardi Equations and Gauss Equations 2:

From the first set of Gauss equations we know,

$$\overrightarrow{\Phi}_{uu} = \Gamma_{11}^1\overrightarrow{\Phi}_u + \Gamma_{11}^2\overrightarrow{\Phi}_v + L\overrightarrow{N}.$$

Differentiate both sides with respect to v:

$$\left(\overrightarrow{\Phi}_{uu}\right)_v = \Gamma_{11}^1\overrightarrow{\Phi}_{uv} + (\Gamma_{11}^1)_v\overrightarrow{\Phi}_u + \Gamma_{11}^2\overrightarrow{\Phi}_{vv} + (\Gamma_{11}^2)_v\overrightarrow{\Phi}_v + L\left(\overrightarrow{N}_v\right) + L_v\left(\overrightarrow{N}\right).$$

Similarly, we know from the first set of Gauss equations:

$$\vec{\Phi}_{uv} = \Gamma_{12}^{1}\vec{\Phi}_u + \Gamma_{12}^{2}\vec{\Phi}_v + M\vec{N}.$$

Differentiate both sides with respect to u:

$$\left(\vec{\Phi}_{uv}\right)_u = \Gamma_{12}^{1}\vec{\Phi}_{uu} + (\Gamma_{12}^{1})_u\vec{\Phi}_u + \Gamma_{12}^{2}\vec{\Phi}_{vu} + (\Gamma_{12}^{2})_u\vec{\Phi}_v + M\vec{N}_u + M_u\vec{N}.$$

Since $\left(\vec{\Phi}_{uu}\right)_v = \left(\vec{\Phi}_{uv}\right)_u$ we have:

$$\Gamma_{11}^{1}\vec{\Phi}_{uv} + (\Gamma_{11}^{1})_v\vec{\Phi}_u + \Gamma_{11}^{2}\vec{\Phi}_{vv} + (\Gamma_{11}^{2})_v\vec{\Phi}_v + L\left(\vec{N}_v\right) + L_v(\vec{N})$$
$$= \Gamma_{12}^{1}\vec{\Phi}_{uu} + (\Gamma_{12}^{1})_u\vec{\Phi}_u + \Gamma_{12}^{2}\vec{\Phi}_{vu} + (\Gamma_{12}^{2})_u\vec{\Phi}_v + M\vec{N}_u + M_u\vec{N}.$$

Collecting the $\vec{\Phi}_u, \vec{\Phi}_v$, and $\vec{N}$ terms on one side:

$$((\Gamma_{11}^{1})_v - (\Gamma_{12}^{1})_u)\vec{\Phi}_u + ((\Gamma_{11}^{2})_v - (\Gamma_{12}^{2})_u)\vec{\Phi}_v + (L_v - M_u)\vec{N}$$
$$= \Gamma_{12}^{1}\vec{\Phi}_{uu} + (\Gamma_{12}^{2} - \Gamma_{11}^{1})\vec{\Phi}_{uv} - \Gamma_{11}^{2}\vec{\Phi}_{vv} - L\vec{N}_v + M\vec{N}_u.$$

We can now use the Gauss Equations 1 again on the RHS:

$$= \Gamma_{12}^{1}\left(\Gamma_{11}^{1}\vec{\Phi}_u + \Gamma_{11}^{2}\vec{\Phi}_v + L\vec{N}\right) + (\Gamma_{12}^{2} - \Gamma_{11}^{1})\left(\Gamma_{12}^{1}\vec{\Phi}_u + \Gamma_{12}^{2}\vec{\Phi}_v + M\vec{N}\right)$$
$$- \Gamma_{11}^{2}\left(\Gamma_{22}^{1}\vec{\Phi}_u + \Gamma_{22}^{2}\vec{\Phi}_v + N\vec{N}\right) - L\vec{N}_v + M\vec{N}_u. \qquad (**)$$

Since $\vec{N}\cdot\vec{N} = 1$, we know $\vec{N}\cdot\vec{N}_u = 0$ and $\vec{N}\cdot\vec{N}_v = 0$, so that $\vec{N}_u$ and $\vec{N}_v$ are perpendicular to $\vec{N}$. Equating the $\vec{N}$ components on both sides, we get:

$$L_v - M_u = L\Gamma_{12}^{1} + M(\Gamma_{12}^{2} - \Gamma_{11}^{1}) - N\Gamma_{11}^{2}$$

This is the first Codazzi-Mainardi Equation.

The other Codazzi-Mainardi Equation comes from equating coefficients of $\vec{N}$ in the equation: $\left(\vec{\Phi}_{uv}\right)_v = \left(\vec{\Phi}_{vv}\right)_u$.

To get the 2nd Gauss Equations, recall that if:

$$W\left(\vec{\Phi}_u\right) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a\vec{\Phi}_u + b\vec{\Phi}_v = -\vec{N}_u$$

$$W\left(\vec{\Phi}_v\right) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = c\vec{\Phi}_u + d\vec{\Phi}_v = -\vec{N}_v$$

Then:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \frac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

So we can replace $\vec{N}_u$ and $\vec{N}_v$ with vectors in terms of $\vec{\Phi}_u$ and $\vec{\Phi}_v$ in equation $(**)$. If we then equate the coefficients of $\vec{\Phi}_u$ we get:

$$F\mathrm{K} = (\Gamma_{12}^1)_u - (\Gamma_{11}^1)_v + \Gamma_{12}^2\Gamma_{12}^1 - \Gamma_{11}^2\Gamma_{22}^1$$

The other Gauss 2 Equations are gotten from equating the coefficients of $\vec{\Phi}_v$ in $\left(\vec{\Phi}_{uu}\right)_v = \left(\vec{\Phi}_{uv}\right)_u$ and of $\vec{\Phi}_u$ and $\vec{\Phi}_v$ in $\left(\vec{\Phi}_{uv}\right)_v = \left(\vec{\Phi}_{vv}\right)_u$.

Cor: (Gauss' Theorema Egregium) The Gaussian curvature of a surface is an intrinsic property of the surface, that is, it only depends on the components E, F, and G of the first fundamental form (i.e. the metric tensor) and its higher derivatives.

This means that someone living on the surface could measure the Gaussian curvature of the surface (you don't actually need the components of the unit normal and its derivatives).

Since the Christoffer symbols, $\Gamma^{\mathrm{i}}_{\mathrm{jk}}$, are defined in terms of the components of the first fundamental form and their derivatives, we can use the second set of Gauss equations to find the Gaussian curvature directly from the components of the first fundamental form. This turns out to be the following messy expression:

$$\mathrm{K} = \frac{\begin{vmatrix} -\frac{1}{2}E_{vv}+F_{uv}-\frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u-\frac{1}{2}E_v \\ F_v-\frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}}{(EG-F^2)^2} \ . \quad (***)$$

However, if $F = 0$, we have:

$$\mathrm{K} = -\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right) + \frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)\right].$$

And if $F = 0$ and $E = 1$, we have:

$$\mathrm{K} = -\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial u^2} \ .$$

Ex. Calculate the Gaussian curvature, K, if the first fundamental form is

$$(1+v^2)du^2 + (2uv)dudv + (1+u^2)dv^2.$$

It's messy, but we just need to plug into $(***)$, where:

$$\begin{array}{llll} E = 1+v^2, & E_u = 0, & E_v = 2v, & E_{vv} = 2 \\ F = uv, & F_u = v, & F_v = u, & F_{uv} = 1 \\ G = 1+u^2, & G_u = 2u, & G_v = 0, & G_{uu} = 2. \end{array}$$

$$\mathrm{K} = \frac{\begin{vmatrix} -1 & 0 & 0 \\ 0 & 1+v^2 & uv \\ 0 & uv & 1+u^2 \end{vmatrix} - \begin{vmatrix} 0 & v & u \\ v & 1+v^2 & uv \\ u & uv & 1+u^2 \end{vmatrix}}{((1+u^2)(1+v^2)-u^2v^2)^2}$$

$$\mathrm{K} = \frac{-[(1+v^2)(1+u^2)-u^2v^2] - \left[-v(v(1+u^2)-u^2v) + u\left(uv^2 - u(1+v^2)\right)\right]}{(1+u^2+v^2)^2}$$

$$\mathrm{K} = \frac{-1}{(1+u^2+v^2)^2}.$$

<u>The Fundamental Theorem of Surface Theory</u>:

If E, F, G and L, M, N are smooth functions of (u,v) that satisfy the Codazzi-Mainardi and Gauss 2 equations and $EG - F^2 > 0$, then there exists a parametrization, $\vec{\Phi}(u,v)$, of a regular orientable surface such that the first fundamental form of $\vec{\Phi}$ is $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ and the second fundamental form of $\vec{\Phi}$ is $\begin{pmatrix} L & M \\ M & N \end{pmatrix}$. Furthermore, this surface is uniquely determined up to rotation and translation in $\mathbb{R}^3$.

By Gauss' theorema egregium, if $F_1 = \bar{F}_1$, where $F_1, \bar{F}_1$ are first fundamental forms of surface patches on two different surfaces, then the Gauss curvatures at every point in the surface patches must be equal, i.e., $\mathrm{K} = \overline{\mathrm{K}}$. However, the converse is not true. You can have two surfaces where the Gaussian curvatures are equal at every point, but the first fundamental forms are not, i.e., $\mathrm{K} = \overline{\mathrm{K}}$ but $F_1 \neq \bar{F}_1$ (see HW problem).

Suppose two surfaces have equal first fundamental forms at every point does that mean the the second fundamental forms must also be equal? The answer is no. For example, the standard parametrizations of a cylinder in $\mathbb{R}^3$ given by $x^2 + y^2 = 1$ and the standard parametrization of the x-y plane hve the same first fundamental forms but different second fundamental forms.

Ex. Show there is no surface patch whose first and second fundamental forms are:

$$F_1 = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u \end{pmatrix} \qquad \text{i.e.,} \qquad du^2 + (\cos^2 u)\, dv^2$$

$$F_2 = \begin{pmatrix} \cos^2 u & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{i.e.,} \qquad (\cos^2 u)\, du^2 + dv^2.$$

By the fundamental theorem of surface theory, E, F, G and L, M, N must satisfy the Codazzi-Mainardi and Gauss 2 equations.

The Codazzi-Mainardi equations say:

$$L_v - M_u = L(\Gamma_{12}^1) + M(\Gamma_{12}^2 - \Gamma_{11}^1) - N(\Gamma_{11}^2)$$

$$M_v - N_u = L(\Gamma_{22}^1) + M(\Gamma_{22}^2 - \Gamma_{12}^1) - N(\Gamma_{12}^2).$$

We can calculate the Christoffel symbols, $\Gamma^{\mathrm{i}}_{\mathrm{jk}}$, from the components of the first fundamental form, E, F, and G:

$$E = 1, \qquad F = 0, \qquad G = \cos^2 u.$$

For example: $\Gamma^1_{11} = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)} = \frac{(\cos^2 u)(0) - 2(0)(0) + 0(0)}{2(\cos^2 u - 0)} = 0.$

In fact, the only non-zero Christoffel symbol is:

$$\Gamma^1_{22} = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)} = \frac{-(\cos^2 u)(-2\cos u \sin u)}{2\cos^2 u} = (\cos u)\sin u.$$

The second Codazzi-Mainardi equation is not satisfied because given

$L = \cos^2 u\,,\ M = 0,\ N = 1 \ \Longrightarrow\ M_v - N_u = 0$, but:

$$L(\Gamma^1_{22}) + M(\Gamma^2_{22} - \Gamma^1_{12}) - N(\Gamma^2_{12}) = (\cos^2 u)(\cos u)(\sin u) \neq 0.$$

Thus, there is no surface with those first and second fundamental forms.