Gaussian Curvature and Mean Curvature

Def: Let W be the Weingarten map (i.e., $W = -D_p\tilde{G}$, where $\tilde{G}$ is the Gauss map) of an orientated surface at a point, $p \in S$. The **Gauss curvature** K and **mean curvature** H of S at p are defined by:

$$\mathrm{K} = \det(W_p), \qquad H = \frac{1}{2}\operatorname{trace}(W_p).$$

Recall that the determinant and trace of any linear transformation (like W) can be computed as the determinant and the sum of the diagonal elements of the matrix of the linear transformation with respect to any basis. That is, the determinant and trace of a linear transformation are unchanged (or invariant) when you change the basis of the linear space.

If the sign of the unit normal of S is changed then the sign of W also changes (i.e. the matrix representation of W get multiplied by -1), thus $\mathrm{K} = \det(W)$ is unchanged but $\operatorname{trace}(W)$ gets multiplied by -1. Thus, K is uniquely defined at $p \in S$ even if S is a nonorientable surface. However, H is only uniquely defined on an orientable surface.

To get an explicit formula for H and K, start with a coordinate patch, $\vec{\Phi}$ of S, $\vec{\Phi}: U \subseteq \mathbb{R}^2 \to S$. Then the first and second fundamental forms can be represented by:

$$E\,du^2 + 2F\,dudv + G\,dv^2 \qquad\qquad L\,du^2 + 2M\,dudv + N\,dv^2$$

$$F_1 = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \qquad\qquad F_2 = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

$$E = \vec{\Phi}_u \cdot \vec{\Phi}_u \qquad\qquad L = \vec{\Phi}_{uu} \cdot \vec{N}$$

$$F = \vec{\Phi}_u \cdot \vec{\Phi}_v = \vec{\Phi}_v \cdot \vec{\Phi}_u \qquad\qquad M = \vec{\Phi}_{uv} \cdot \vec{N} = \vec{\Phi}_{vu} \cdot \vec{N}$$

$$G = \vec{\Phi}_v \cdot \vec{\Phi}_v \qquad\qquad N = \vec{\Phi}_{vv} \cdot \vec{N}.$$

Proposition: The matrix, $W_p: T_p(S) \to T_p(S)$, with respect to the basis $\{\vec{\Phi}_u, \vec{\Phi}_v\}$ of T_pS is $F_1^{-1}F_2$.

Proof: We know the following:

$$D_p\tilde{G}(\vec{\Phi}_u) = \vec{N}_u$$

$$D_p\tilde{G}(\vec{\Phi}_v) = \vec{N}_v$$

$$W_p = -D_p(\tilde{G})$$

$$W_p(\vec{\Phi}_u) = -\vec{N}_u$$

$$W_p(\vec{\Phi}_v) = -\vec{N}_v \,.$$

So if the matrix for W_p is $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ with respect to the basis $\{\vec{\Phi}_u, \vec{\Phi}_v\}$, then:

$$W_p(\vec{\Phi}_u) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a\vec{\Phi}_u + b\vec{\Phi}_v = -\vec{N}_u$$

$$W_p(\vec{\Phi}_v) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = c\vec{\Phi}_u + d\vec{\Phi}_v = -\vec{N}_v \,.$$

Now take the dot product of each equation with $\overrightarrow{\Phi}_u$ and $\overrightarrow{\Phi}_v$ and use:
$\vec{N}_u \cdot \overrightarrow{\Phi}_u = -L$, $\vec{N}_u \cdot \overrightarrow{\Phi}_v = \vec{N}_v \cdot \overrightarrow{\Phi}_u = -M$, and $\vec{N}_v \cdot \overrightarrow{\Phi}_v = -N$ to get:

1) $a\overrightarrow{\Phi}_u \cdot \overrightarrow{\Phi}_u + b\overrightarrow{\Phi}_v \cdot \overrightarrow{\Phi}_u = -\vec{N}_u \cdot \overrightarrow{\Phi}_u$

$$aE + bF = L$$

2) $c\overrightarrow{\Phi}_u \cdot \overrightarrow{\Phi}_u + d\overrightarrow{\Phi}_v \cdot \overrightarrow{\Phi}_u = -\vec{N}_v \cdot \overrightarrow{\Phi}_u$

$$cE + dF = M$$

3) $a\overrightarrow{\Phi}_u \cdot \overrightarrow{\Phi}_v + b\overrightarrow{\Phi}_u \cdot \overrightarrow{\Phi}_v = -\vec{N}_u \cdot \overrightarrow{\Phi}_v$

$$aF + bG = M$$

4) $c\overrightarrow{\Phi}_u \cdot \overrightarrow{\Phi}_v + d\overrightarrow{\Phi}_v \cdot \overrightarrow{\Phi}_v = -\vec{N}_v \cdot \overrightarrow{\Phi}_v$

$$cF + dG = N.$$

So we can write these four equations in four unknowns as:

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

That is,

$$F_2 = F_1 \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Hence the matrix for W_p, with respect to the basis $\{\overrightarrow{\Phi}_u, \overrightarrow{\Phi}_v\}$ is:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = F_1^{-1} F_2 \,.$$

Cor. $H = \frac{LG-2MF+NE}{2(EG-F^2)}$; $\mathrm{K} = \frac{LN-M^2}{EG-F^2}$

Proof:

$$\mathrm{K} = \det(W_p) = \det(F_1^{-1}F_2)$$
$$= \frac{\det(F_2)}{\det(F_1)}$$
$$= \frac{LN-M^2}{EG-F^2}.$$

Note: $\det(F_1) > 0$ since $\det(F_1) = \left\|\vec{\Phi}_u \times \vec{\Phi}_v\right\|^2$ and $\vec{\Phi}$ is regular. Thus the sign of the Gauss curvature $\mathrm{K} = \frac{\det(F_2)}{\det(F_1)}$ is the same as the sign of $\det(F_2)$. So elliptic points have $\mathrm{K} > 0$, hyperbolic points have $\mathrm{K} < 0$, and parabolic or planar points have $\mathrm{K} = 0$.

To compute H we have:

$$F_1^{-1}F_2 = \frac{1}{\det(F_1)}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

$$F_1^{-1}F_2 = \frac{1}{EG-F^2}\begin{pmatrix} LG-MF & MG-NF \\ ME-LF & NE-MF \end{pmatrix}.$$

$$H = \frac{1}{2}\text{trace}\,(F_1^{-1}F_2)$$
$$= \frac{LG-2MF+NE}{2(EG-F^2)}.$$

Ex. Calculate the Weingarten map W, the Gaussian and the mean curvature of the sphere of radius R given by:

$$\vec{\Phi}(\phi,\theta) = (R\cos\theta\sin\phi\,, R\sin\theta\sin\phi\,, R\cos\phi).$$

We know from earlier calculations:

$$F_1 = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\phi \end{pmatrix}$$

$$F_2 = \begin{pmatrix} -R & 0 \\ 0 & -R\sin^2\phi \end{pmatrix}$$

And that

$$W = F_1^{-1}F_2\ .$$

$$F_1^{-1} = \frac{1}{R^4\sin^2\phi}\begin{pmatrix} R^2\sin^2\phi & 0 \\ 0 & R^2 \end{pmatrix}$$

$$W = \frac{1}{R^4\sin^2\phi}\begin{pmatrix} R^2\sin^2\phi & 0 \\ 0 & R^2 \end{pmatrix}\begin{pmatrix} -R & 0 \\ 0 & -R\sin^2\phi \end{pmatrix}$$

$$W = \begin{pmatrix} -\frac{1}{R} & 0 \\ 0 & -\frac{1}{R} \end{pmatrix}.$$

The easiest way to get the Gaussian curvature and the mean curvature given we have an expression for the Weingarten map W, is to say:

$$\mathrm{K} = \det(W) = \frac{1}{R^2}, \quad H = \frac{1}{2}\big(trace(W)\big) = -\frac{1}{R}.$$

If we hadn't calculated W we could get the Gaussian and mean curvatures from F_1 and F_2 by:

$$\mathrm{K} = \frac{\det(F_2)}{\det(F_1)} = \frac{R^2 \sin^2\phi}{R^4 \sin^2\phi} = \frac{1}{R^2}$$

$$H = \frac{1}{2}\ \frac{LG - 2MF + NE}{\det(F_1)}$$

where:

$L = -R$ $\qquad\qquad\qquad E = R^2$

$M = 0$ $\qquad\qquad\qquad F = 0$

$N = -R\sin^2\phi$ $\qquad\qquad G = R^2\sin^2\phi$

$$H = \frac{1}{2}\ \frac{(-R)\big(R^2\sin^2\phi\big) - 2(0)(0) + \big(-R\sin^2\phi\big)R^2}{R^4 \sin^2\phi}$$

$$H = \frac{1}{2}\ \frac{-R^3\sin^2\phi - R^3\sin^2\phi}{R^4\sin^2\phi} = -\frac{1}{R}.$$

Ex. Find formulas for the Gaussian curvature and mean curvature of a surface of the form: $\vec{\Phi}(u, v) = (u, v, f(u, v))$, where f is a smooth, real valued function on $U \subseteq \mathbb{R}^2$.

$$K = \frac{\det(F_2)}{\det(F_1)}$$

So, let's find the first and second fundamental forms: F_1, F_2.

$$\vec{\Phi}_u = (1, 0, f_u)$$

$$\vec{\Phi}_v = (0, 1, f_v)$$

$$\vec{\Phi}_u \times \vec{\Phi}_v = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} = -f_u\vec{\imath} - f_v\vec{\jmath} + \vec{k}$$

$$\vec{N} = \frac{(-f_u,\ -f_v,\ 1)}{\sqrt{1+(f_u)^2+(f_v)^2}}$$

$$E = \vec{\Phi}_u \cdot \vec{\Phi}_u = 1 + (f_u)^2$$

$$F = \vec{\Phi}_u \cdot \vec{\Phi}_v = (f_u)(f_v)$$

$$G = \vec{\Phi}_v \cdot \vec{\Phi}_v = 1 + (f_v)^2$$

$$F_1 = \begin{pmatrix} 1 + (f_u)^2 & (f_u)(f_v) \\ (f_u)(f_v) & 1 + (f_v)^2 \end{pmatrix}.$$

$$\vec{\Phi}_{uu} = (0, 0, f_{uu})$$

$$\vec{\Phi}_{uv} = (0, 0, f_{uv})$$

$$\vec{\Phi}_{vv} = (0, 0, f_{vv})$$

$$L = \vec{\Phi}_{uu} \cdot \vec{N} = \frac{f_{uu}}{\sqrt{1+(f_u)^2+(f_v)^2}}$$

$$M = \vec{\Phi}_{uv} \cdot \vec{N} = \frac{f_{uv}}{\sqrt{1+(f_u)^2+(f_v)^2}}$$

$$N = \vec{\Phi}_{vv} \cdot \vec{N} = \frac{f_{vv}}{\sqrt{1+(f_u)^2+(f_v)^2}}$$

$$F_2 = \frac{1}{\sqrt{1+(f_u)^2+(f_v)^2}} \begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}.$$

$$\mathrm{K} = \frac{\det(F_2)}{\det(F_1)}$$

$$\mathrm{K} = \frac{\frac{(f_{uu})(f_{vv})-(f_{uv})^2}{1+(f_u)^2+(f_v)^2}}{1+(f_u)^2+(f_v)^2} = \frac{(f_{uu})(f_{vv})-(f_{uv})^2}{(1+(f_u)^2+(f_v)^2)^2}.$$

$$H = \frac{1}{2}\left(\frac{LG-2MF+NE}{\det(F_1)}\right)$$

$$H = \frac{1}{2}\left(\frac{(f_{uu})\left(1+(f_v)^2\right)-2(f_u)(f_v)(f_{uv})+f_{vv}\left(1+(f_u)^2\right)}{(1+(f_u)^2+(f_v)^2)^{\frac{3}{2}}}\right).$$

Ex. Find the Gaussian and mean curvature of the paraboloid:

$$\vec{\Phi}(u,v) = (u, v, u^2 + v^2)$$

using the formulas for K and H for the surface of the form:

$$\vec{\Phi}(u,v) = \big(u, v, f(u,v)\big) \text{ with } f(u,v) = u^2 + v^2.$$

$$f_u = 2u$$

$$f_v = 2v$$

$$f_{uu} = 2$$

$$f_{uv} = 0$$

$$f_{vv} = 2$$

$$\mathrm{K} = \frac{(f_{uu})(f_{vv}) - (f_{uv})^2}{(1+(f_u)^2+(f_v)^2)^2}$$

$$\mathrm{K} = \frac{4}{(1+4u^2+4v^2)^2} > 0.$$

$$H = \frac{1}{2}\left(\frac{(f_{uu})\big(1+(f_v)^2\big) - 2(f_u)(f_v)(f_{uv}) + f_{vv}\big(1+(f_u)^2\big)}{(1+(f_u)^2+(f_v)^2)^{\frac{3}{2}}}\right)$$

$$= \frac{1}{2}\left(\frac{2\big(1+4v^2\big)+2\big(1+4u^2\big)}{(1+4u^2+4v^2)^{\frac{3}{2}}}\right)$$

$$H = \frac{2+4u^2+4v^2}{(1+4u^2+4v^2)^{\frac{3}{2}}} > 0.$$

Ex. Find the Gaussian and mean curvature of the hyperbolic paraboloid given by:

$$\vec{\Phi}(u,v) = (u, v, u^2 - v^2).$$

$f(u,v) = u^2 - v^2$

$f_u = 2u$

$f_v = -2v$

$f_{uu} = 2$

$f_{uv} = 0$

$f_{vv} = -2$

$$\mathrm{K} = \frac{-4}{(1+4u^2+4v^2)^2} < 0$$

$$H = \frac{1}{2}\left(\frac{2(1+4v^2)-2(1+4u^2)}{(1+4u^2+4v^2)^{\frac{3}{2}}}\right)$$

$$H = \frac{4v^2-4u^2}{(1+4u^2+4v^2)^{\frac{3}{2}}}.$$

$H > 0$ when $v^2 > u^2$ or $|v| > |u|$

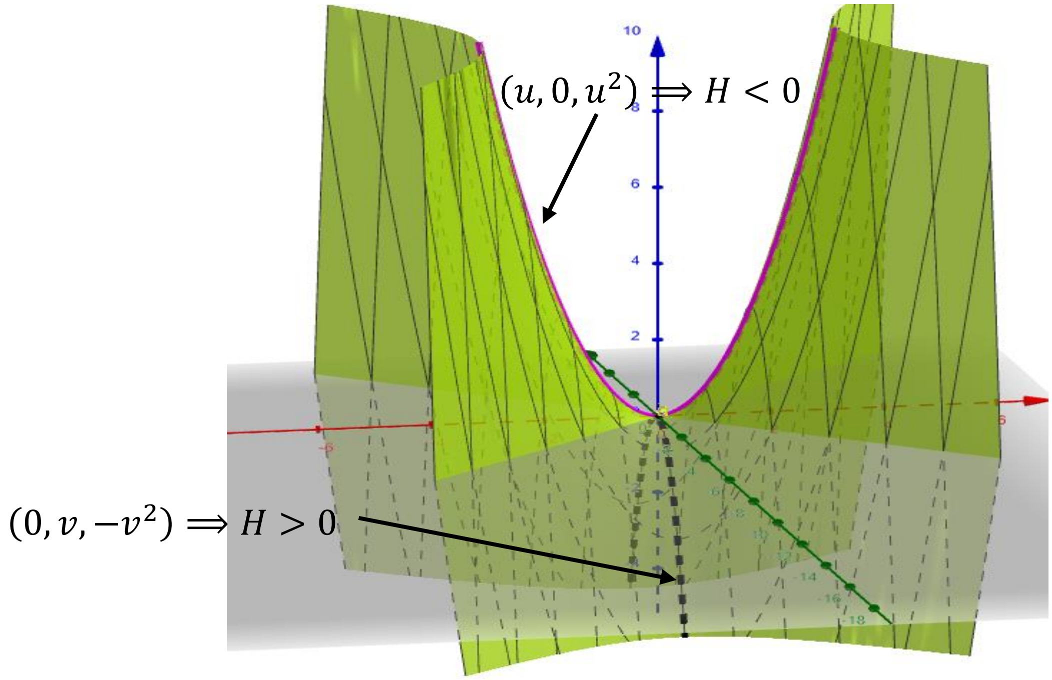
$H = 0$ when $v^2 = u^2$ or $v = \pm u$

$H < 0$ when $|v| < |u|$

For example,

when $u = 0, v \neq 0$; $\vec{\Phi}(0,v) = (0, v, -v^2) \Longrightarrow H > 0$

when $v = 0, u \neq 0$; $\vec{\Phi}(u,0) = (u, 0, u^2) \Longrightarrow H < 0.$



We have the following geometric interpretation of the Gaussian curvature, K.

Theorem: Let $\overrightarrow{\Phi}: U \to \mathbb{R}^3$ be a surface patch such that $(u_0, v_0) \in U$ and

let $\delta > 0$ be such that the closed disc:

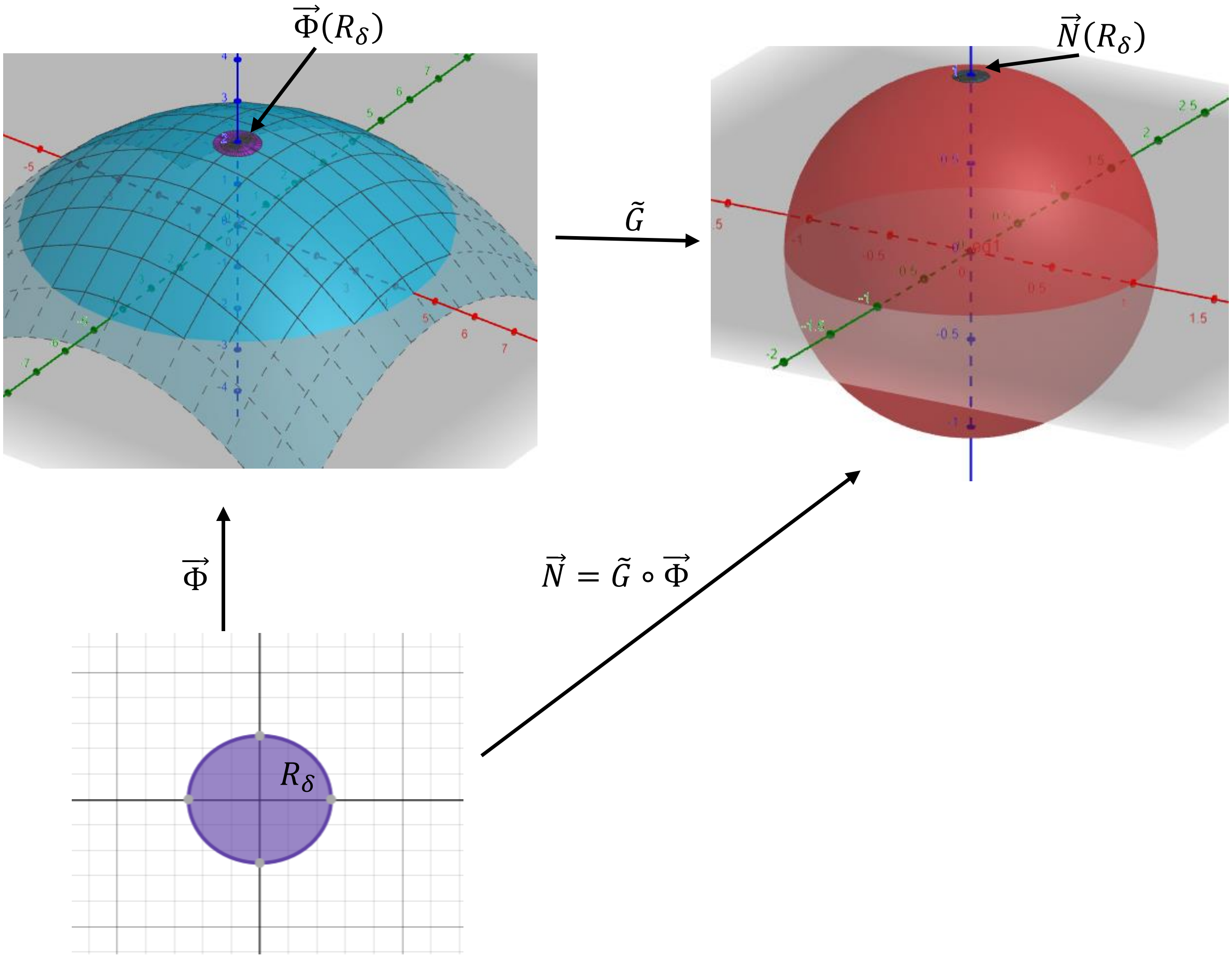
$$R_\delta = \{(u,v) \in \mathbb{R}^2 | (u-u_0)^2 + (v-v_0)^2 \leq \delta\}$$

with center (u_0, v_0) and radius δ is contained in U. Then,

$$\lim_{\delta \to 0} \frac{A\left(\vec{N}(R_\delta)\right)}{A\left(\overrightarrow{\Phi}(R_\delta)\right)} = \left|\mathrm{K}\left(\overrightarrow{\Phi}(u_0, v_0)\right)\right|$$

where $A\left(\vec{N}(R_\delta)\right)$ = area of image of R_δ under $\vec{N} = \tilde{G}(\overrightarrow{\Phi})$

$A\left(\overrightarrow{\Phi}(R_\delta)\right)$ = area of image of R_δ under $\overrightarrow{\Phi}$.



Proof: $$A\left(\vec{N}(R_\delta)\right) = \iint_{R_\delta} \left\|\vec{N}_u \times \vec{N}_v\right\| \, dudv$$

$$A\left(\vec{\Phi}(R_\delta)\right) = \iint_{R_\delta} \left\|\vec{\Phi}_u \times \vec{\Phi}_v\right\| \, dudv$$

We saw earlier that:

$$W_p = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\vec{N}_u = -a\vec{\Phi}_u - b\vec{\Phi}_v$$

$$\vec{N}_v = -c\vec{\Phi}_u - d\vec{\Phi}_v$$

$$\vec{N}_u \times \vec{N}_v = (a\vec{\Phi}_u + b\vec{\Phi}_v) \times (c\vec{\Phi}_u + d\vec{\Phi}_v)$$
$$= (ad - bc)\vec{\Phi}_u \times \vec{\Phi}_v$$
$$= (\mathrm{K})\vec{\Phi}_u \times \vec{\Phi}_v \qquad \text{since } \mathrm{K} = \det(W_p).$$

$$\frac{A\left(\vec{N}(R_\delta)\right)}{A\left(\vec{\Phi}(R_\delta)\right)} = \frac{\iint_{R_\delta} \|\vec{N}_u \times \vec{N}_v\| dudv}{\iint_{R_\delta} \|\vec{\Phi}_u \times \vec{\Phi}_v\| dudv}$$

$$= \frac{\iint_{R_\delta} |\mathrm{K}| \|\vec{\Phi}_u \times \vec{\Phi}_v\| dudv}{\iint_{R_\delta} \|\vec{\Phi}_u \times \vec{\Phi}_v\| dudv}.$$

Since $\mathrm{K}\ (u, v)$ is a continuous function, given any $\epsilon > 0$, there exists a $\delta > 0$ such that: $|\mathrm{K}(u, v) - \mathrm{K}(u_0, v_0)| < \epsilon$, if $(u, v) \in R_\delta$.

By the triangle inequality:

$$\big||a| - |b|\big| \leq |a - b|.$$

So it follows that:

$$\big||\mathrm{K}(u, v)| - |\mathrm{K}(u_0, v_0)|\big| \leq |\mathrm{K}(u, v) - \mathrm{K}(u_0, v_0)| < \epsilon$$

if $(u, v) \in R_\delta$.

Thus, if $(u, v) \in R_\delta$, then:

$$-\epsilon < |\mathrm{K}(u,v)| - \ |\mathrm{K}(u_0,v_0)| < \epsilon$$

$$|(\mathrm{K}(u_0,v_0)| - \epsilon < |\mathrm{K}(u,v)| < |\mathrm{K}(u_0,v_0)| + \epsilon$$

Multiplying through by $\left\|\vec{\Phi}_u \times \vec{\Phi}_v\right\|$ and integrating over R_δ:

$$(|\mathrm{K}(u_0,v_0)| - \epsilon) \iint_{R_\delta} \left\|\vec{\Phi}_u \times \vec{\Phi}_v\right\| < \iint_{R_\delta} |\mathrm{K}(u,v)| \left\|\vec{\Phi}_u \times \vec{\Phi}_v\right\| du\, dv$$

$$< (|\mathrm{K}(u_0,v_0)| + \epsilon) \iint_{R_\delta} \left\|\vec{\Phi}_u \times \vec{\Phi}_v\right\| dudv\ .$$

Now divide the entire inequality by $\iint_{R_\delta} \left\|\vec{\Phi}_u \times \vec{\Phi}_v\right\|$:

$$\mathrm{K}|(u_0,v_0)| - \epsilon < \frac{A\left(\vec{N}(R_\delta)\right)}{A\left(\vec{\Phi}(R_\delta)\right)} < |\mathrm{K}(u_0,v_0)| + \epsilon$$

$$\left|\frac{A\left(\vec{N}(R_\delta)\right)}{A\left(\vec{\Phi}(R_\delta)\right)} - |\mathrm{K}(u_0,v_0)|\right| < \epsilon$$

$$\lim_{\delta \to 0} \frac{A\left(\vec{N}(R_\delta)\right)}{A\left(\vec{\Phi}(R_\delta)\right)} = |\mathrm{K}(u_0,v_0)|.$$

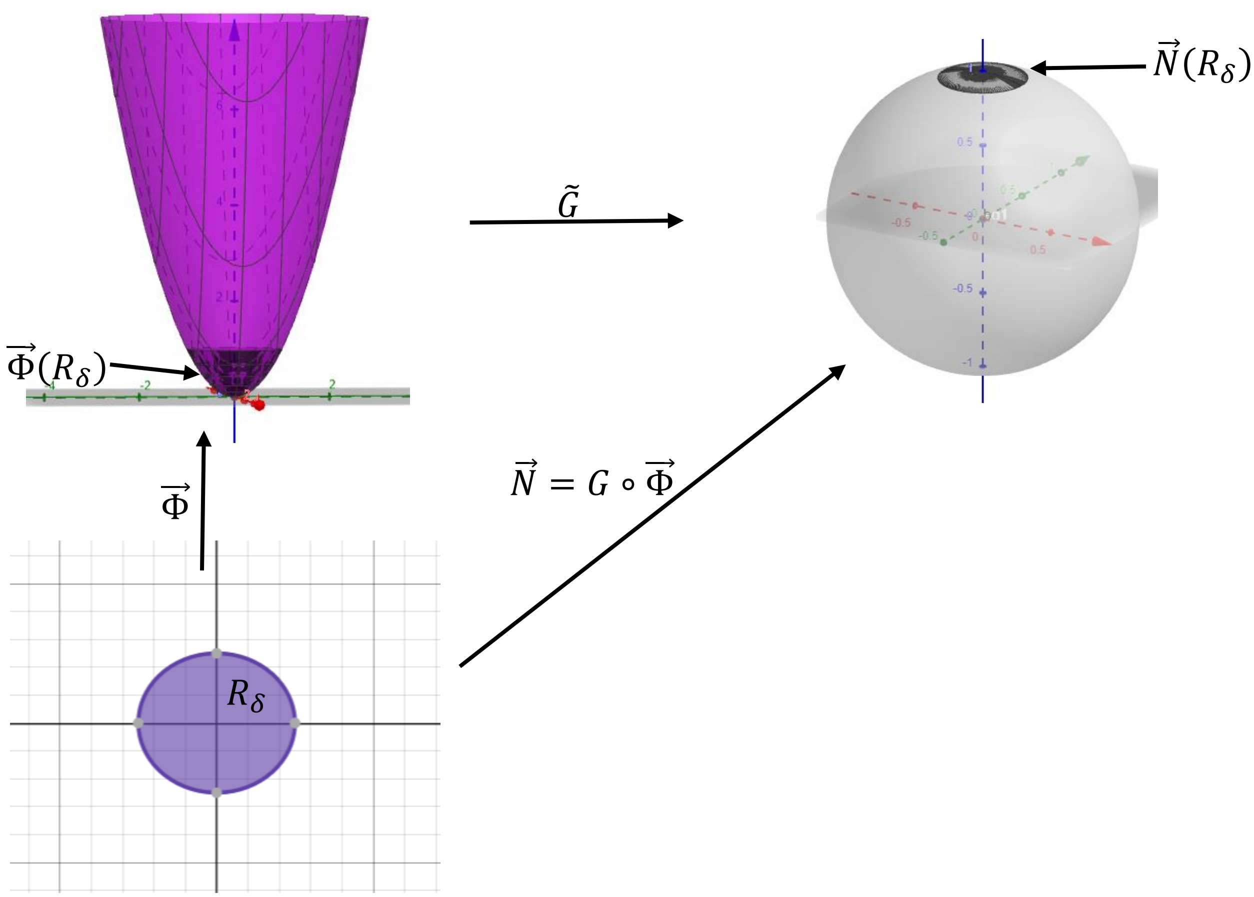
The image of the Gauss map for a plane is a single point (the unit normal for a plane is the same at every point) and for a right circular cylinder is a circle. In both cases, the area of the image of a region around a point on the surface is 0. Thus, the Guassian curvature is 0 for a plane or the right circular cylinder.

Ex. Consider the paraboloid parametrized by $\vec{\Phi}(u,v) = (u, v, u^2 + v^2)$. We saw earlier that the Gauss curvature is given by $\mathrm{K} = \frac{4}{(1+4u^2+4v^2)^2} > 0$.

Thus, in particular, $\mathrm{K}(\vec{\Phi}(0,0)) = 4$. Show by direct calculation that:

$$\lim_{\delta \to 0} \frac{A\left(\vec{N}(R_\delta)\right)}{A\left(\vec{\Phi}(R_\delta)\right)} = \left|\mathrm{K}\left(\vec{\Phi}(0,0)\right)\right| = 4$$

where R_δ is the disk of radius δ about the point $(0,0)$ in the u-v plane.



By direct calculation we get: $\vec{\Phi}_u \times \vec{\Phi}_v = (-2u, -2v, 1)$ and

$$\left\|\vec{\Phi}_u \times \vec{\Phi}_v\right\| = \sqrt{1 + 4u^2 + 4v^2}\,.$$

Thus we have:

$$A\left(\vec{\Phi}(R_\delta)\right) = \iint_{u^2+v^2\leq\delta^2} \sqrt{1 + 4u^2 + 4v^2}\, dudv$$

Now change to polar coordinates:

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{\delta} \sqrt{1 + 4r^2}\, (r) dr d\theta$$

$$= \frac{\pi}{6}\left[(1 + 4\delta^2)^{\frac{3}{2}} - 1\right].$$

Again by direct calculation we get:

$$\vec{N}(u, v) = \frac{1}{\sqrt{1+4u^2+4v^2}}(-2u, -2v, 1)$$

$$\vec{N}_u(u, v) = \frac{1}{(1+4u^2+4v^2)^{\frac{3}{2}}}(-2 - 8v^2, 8uv, -4u)$$

$$\vec{N}_v(u, v) = \frac{1}{(1+4u^2+4v^2)^{\frac{3}{2}}}(\, 8uv, -2 - 8u^2, -4v)$$

$$\left\|\vec{N}_u \times \vec{N}_v\right\| = \frac{4}{(1+4u^2+4v^2)^{\frac{3}{2}}}$$

Thus we have:

$$A\left(\vec{N}(R_\delta)\right) = \iint_{u^2+v^2\le\delta^2} \frac{4}{(1+4u^2+4v^2)^{\frac{3}{2}}}\, dudv.$$

Changing to polar coordinates we get:

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{\delta}\left(\frac{4}{(1+4r^2)^{\frac{3}{2}}}\right)(r)drd\theta$$

$$= -2\pi\left(\frac{1}{\sqrt{1+4\delta^2}} - 1\right).$$

So we have:

$$\lim_{\delta\to 0}\frac{A\left(\vec{N}(R_\delta)\right)}{A\left(\vec{\Phi}(R_\delta)\right)} = \lim_{\delta\to 0} -\frac{2\pi\left(\frac{1}{\sqrt{1+4\delta^2}}-1\right)}{\frac{\pi}{6}\left[(1+4\delta^2)^{\frac{3}{2}}-1\right]} = \lim_{\delta\to 0} -\frac{12\left(\frac{1}{\sqrt{1+4\delta^2}}-1\right)}{\left[(1+4\delta^2)^{\frac{3}{2}}-1\right]}.$$

Now let $t = \sqrt{1+4\delta^2}$; so as $\delta \to 0$, $t \to 1$.

$$= \lim_{t\to 1} -\frac{12\left(\frac{1}{t}-1\right)}{[t^3-1]}\left[\frac{t}{t}\right] = \lim_{t\to 1} -\frac{12(1-t)}{[t^4-t]}$$

$$= \lim_{t\to 1} -\frac{-12}{4t^3-1} = 4.$$