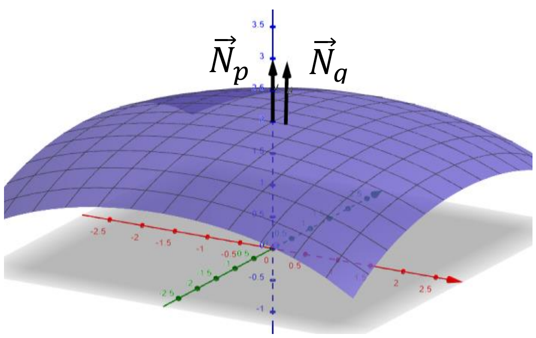
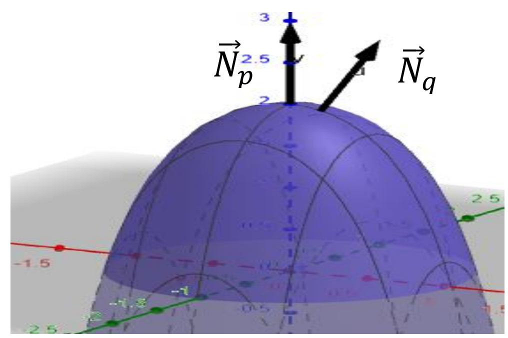
The Gauss and Weingarten Maps

A second approach to defining curvature of an **oriented surface** (a surface is orientable if given any two coordinate patches $\vec{\Phi}_i: U_i \to S,\ \vec{\Phi}_j: U_j \to S$ then $\vec{N}_{p,i} = \vec{N}_{p,j}$ for any $p \in U_i \cap U_j$) is to consider its unit normal, $\vec{N}$. The way $\vec{N}$ varies as we move to nearby points on the surface reflects the curvature of S. If $\vec{N}$ varies "slowly", then the curvature is "small." If $\vec{N}$ varies "quickly", then the curvature is "large."





We can define a mapping of a smooth, regular surface, S, into the unit sphere, S^2, by:

$$\tilde{G}: S \to S^2$$

$$p \to \vec{N}_p = \text{unit normal at } p \in S.$$

Since $\vec{N}_p$ is a unit vector, it represents a point in S^2.

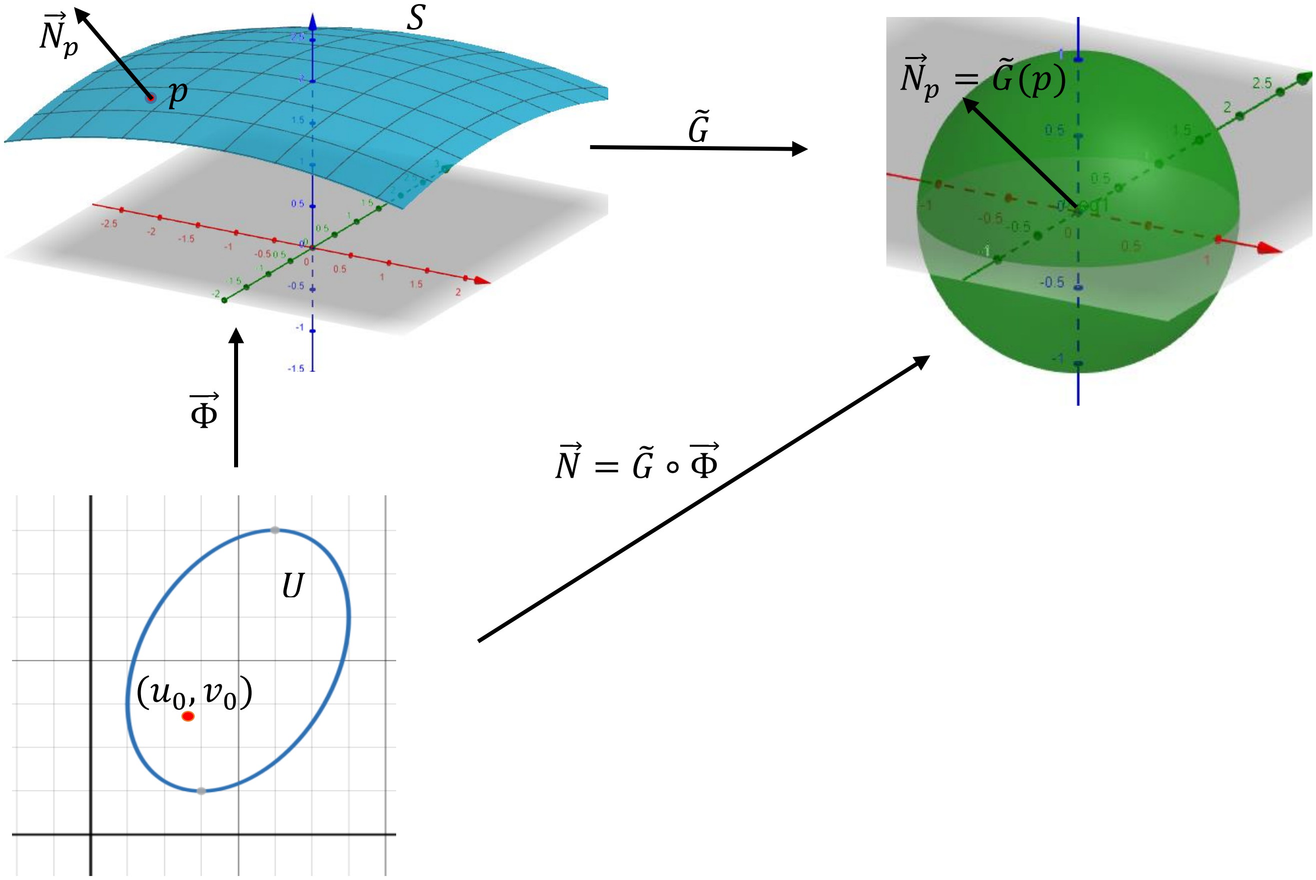
$\tilde{G}$ is called the **Gauss map**.

In practice we calculate this as follows: if $p = \vec{\Phi}(u_0, v_0)$, then

$$\tilde{G}\left(\vec{\Phi}(u_0, v_0)\right) = \frac{(\vec{\Phi}_u \times \vec{\Phi}_v)(u_0, v_0)}{\|(\vec{\Phi}_u \times \vec{\Phi}_v)(u_0, v_0)\|}.$$

If $\overrightarrow{\Phi}: U \subseteq \mathbb{R}^2 \rightarrow S$ is a coordinate patch for S and $\tilde{G}: S \rightarrow S^2$ is the Gauss map, then $\tilde{G} \circ \overrightarrow{\Phi}: U \subseteq \mathbb{R}^2 \rightarrow S^2$.

Let's call $\tilde{G} \circ \overrightarrow{\Phi}$ the mapping $\vec{N}: U \subseteq \mathbb{R}^2 \rightarrow S^2$. Thus if $\overrightarrow{\Phi}(u_0, v_0) = p \in S$, then $\vec{N}(u_0, v_0) = \vec{N}_p$, the unit normal to the surface S at p.



Notice that for all $(u, v) \in U$, $\vec{N}(u, v) \cdot \vec{N}(u, v) = 1$. Differentiating this equation with respect to u and v we get:

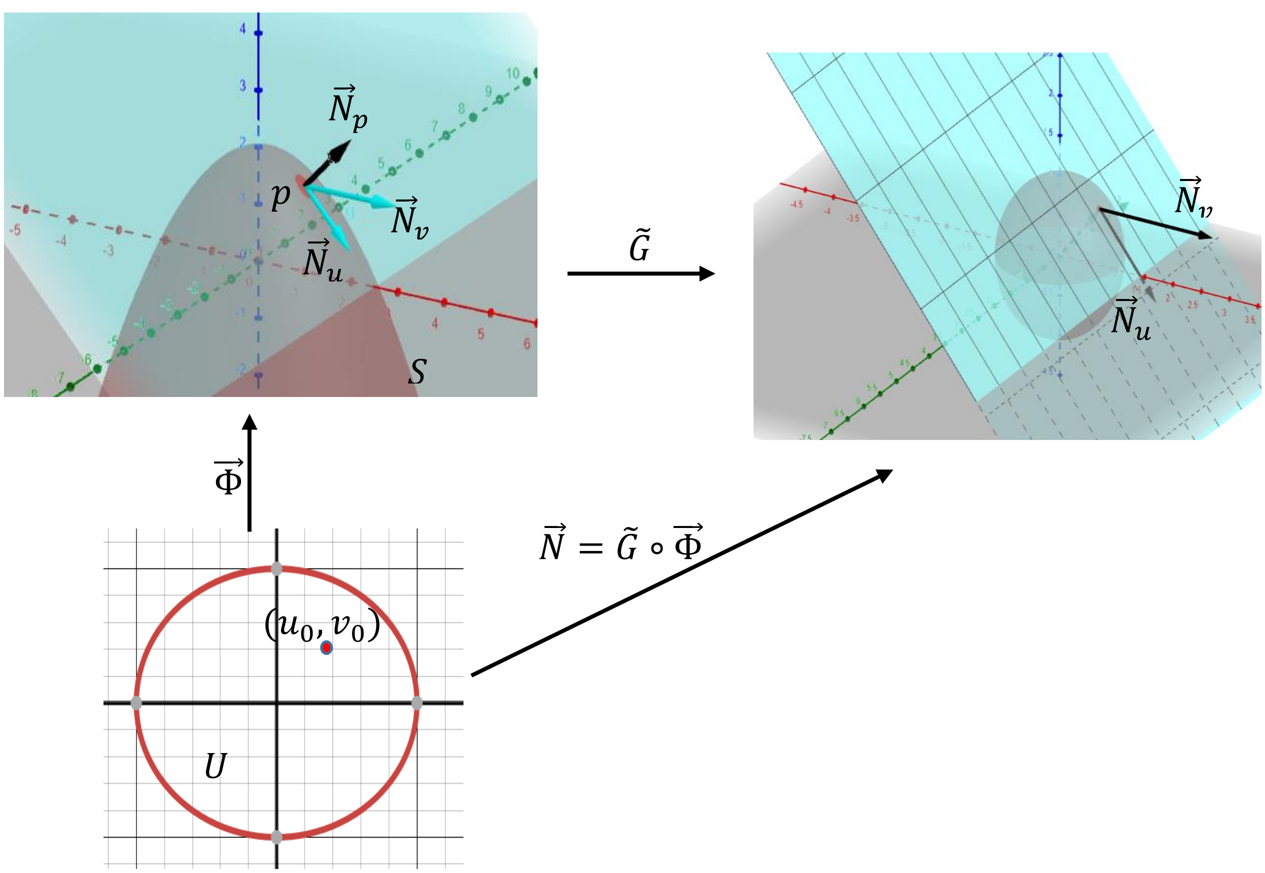
$$\vec{N}_u \cdot \vec{N} + \vec{N} \cdot \vec{N}_u = 0 \quad \text{or} \quad \vec{N} \cdot \vec{N}_u = 0 \qquad (*)$$

Similarly:

$$\vec{N} \cdot \vec{N}_v = 0 \qquad (**)$$

In particular, if $\vec{\Phi}(u_0, v_0) = p \in S$ and $\tilde{G}(p) = q \in S^2$, then $\vec{N}(u_0, v_0) = q \in S^2$. Since $\vec{N}(u_0, v_0)$ is the unit normal to S at p, equations $(*)$ and $(**)$ say that if the vectors $\vec{N}_u(u_0, v_0)$ and $\vec{N}_v(u_0, v_0)$ are each non-zero then they are perpendicular to the unit normal to S at p.

Hence, both $\vec{N}_u(u_0, v_0)$ and $\vec{N}_v(u_0, v_0)$ lie in the tangent plane to S at p, T_pS. But $\vec{N}: U \subseteq \mathbb{R}^2 \to S^2$ is a parametrization for a subset of S^2. Thus, $\vec{N}_u(u_0, v_0)$ and $\vec{N}_v(u_0, v_0)$ lie in the tangent plane of S^2 at q.



Assuming that $\vec{N}_u(u_0, v_0), \vec{N}_v(u_0, v_0)$ together span the tangent plane of S at p and the tangent plane of S^2 at $\tilde{G}(p) = q$:

$$T_pS = T_{\tilde{G}(p)}(S^2)$$

i.e. they are the same plane.

The rate at which the unit normal to S at p, $\vec{N}_p$, varies is measured by the derivative (or differential) of $\tilde{G}$.

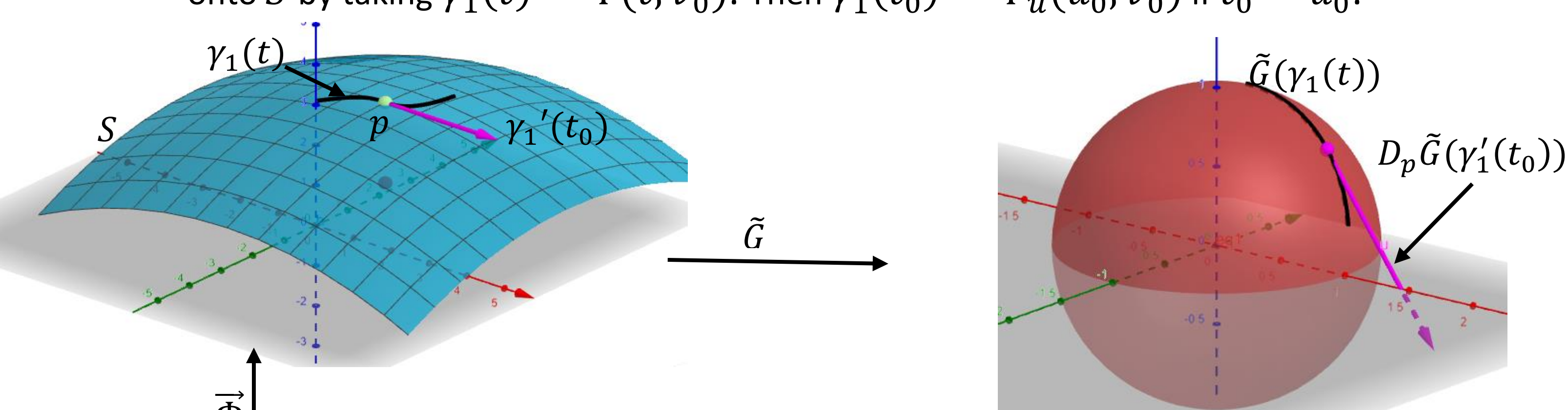
$$D_p\tilde{G}: T_pS \to T_{\tilde{G}(p)}(S^2).$$

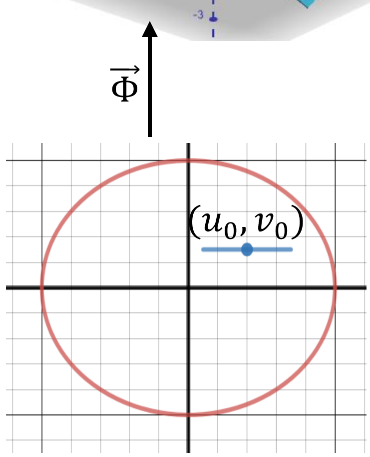
As just noted, $T_pS = T_{\tilde{G}(p)}(S^2)$ so we can think of $D_p\tilde{G}$ as mapping $T_p(S)$ into $T_p(S)$. As we know, given a vector $\vec{w} \in T_pS$ we define:

$$D_p\tilde{G}(\vec{w}) = \vec{\vec{w}} \in T_{\tilde{G}(p)}(S^2) = T_pS$$

by taking any curve, γ on S, passing through $p \in S$ (i.e. $\gamma(t_0) = p$) with $\gamma'(t_0) = \vec{w}$, then $\vec{\vec{w}} = \left(\tilde{G} \circ \gamma\right)'(t_0)$.

In particular, take the curve in U defined by (t, v_0) and then project it onto S by taking $\gamma_1(t) = \vec{\Phi}(t, v_0)$. Then $\gamma_1'(t_0) = \vec{\Phi}_u(u_0, v_0)$ if $t_0 = u_0$.





$$\gamma_1'(t) = \vec{\Phi}_u \frac{du}{dt} + \vec{\Phi}_v \frac{dv}{dt} = \vec{\Phi}_u(u_0, v_0).$$

Then: $D_p\tilde{G}\left(\vec{\Phi}_u(u_0,v_0)\right)=\frac{d}{dt}\left((\tilde{G}\circ\vec{\Phi})(t,v_0)\right)\Big|_{t=t_0}=\frac{d}{dt}\left((\vec{N})(t,v_0)\right)\Big|_{t=t_0}$

$$=\vec{N}_u\frac{d(t)}{dt}\Big|_{t=t_0}+\vec{N}_v\frac{d}{dt}(\text{constant})$$

$$D_p\tilde{G}\left(\vec{\Phi}_u(u_0,v_0)\right)=\vec{N}_u(u_0,v_0).$$

Similarly, take the curve in U defined by (u_0,t) and project it onto S by $\gamma_2(t)=\vec{\Phi}(u_0,t)$. Then, $\gamma_2'(t_0)=\vec{\Phi}_v(u_0,v_0)$ if $t_0=v_0$.

$$D_p\tilde{G}\left(\vec{\Phi}_v(u_0,v_0)\right)=\frac{d}{dt}\left((\tilde{G}\circ\vec{\Phi})(u_0,t)\right)\Big|_{t=t_0}$$

$$=\frac{d}{dt}\left(\vec{N}(u_0,t)\right)\Big|_{t=t_0}=\vec{N}_v(u_0,v_0)$$

$$D_p\tilde{G}\left(\vec{\Phi}_v(u_u,v_u)\right)=\vec{N}_v(u_0,v_0).$$

Hence given any vector $\vec{w}\in T_pS$, we can write it as:

$$\vec{w}=a\vec{\Phi}_u(u_0,v_0)+b\vec{\Phi}_v(u_0,v_0).$$

Thus:

$$D_p\tilde{G}(\vec{w})=D_p\tilde{G}\left(a\vec{\Phi}_u(u_0,v_0)+b\vec{\Phi}_v(u_0,v_0)\right)$$

$$=aD_p\tilde{G}\left(\vec{\Phi}_u(u_0,v_0)\right)+bD_p\tilde{G}\left(\vec{\Phi}_v(u_0,v_0)\right)$$

$$=a\vec{N}_u(u_0,v_0)+b\vec{N}_v(u_0,v_0).$$

Def. Let $p \in S, S$ is a regular smooth surface. The **Weingarten map**, $W_{p,S}$ of S at p, is defined by

$$W_{p,S} = -D_p\tilde{G}$$

(the minus sign will reduce the number of minus signs later).

We want to show that the second fundamental form:

$$L\, du(\vec{w}_1)du(\vec{w}_2) + M\, du(\vec{w}_1)dv(\vec{w}_2) + M\, du(\vec{w}_2)dv(\vec{w}_1) + N\, dv(\vec{w}_1)dv(\vec{w}_2)$$

(where $\vec{w}_1, \vec{w}_2 \in T_pS$) is the same as:

$$< W_{p,S}(\vec{w}_1), \vec{w}_2 > \qquad\qquad (<,> \text{ is the dot product}).$$

To do that we need the following lemma:

Lemma: Let $\vec{\Phi}(u, v)$ be a surface patch with unit normal $\vec{N}(u, v)$, then

$$\vec{N}_u \cdot \vec{\Phi}_u = -L$$

$$\vec{N}_u \cdot \vec{\Phi}_v = \vec{N}_v \cdot \vec{\Phi}_u = -M$$

$$\vec{N}_v \cdot \vec{\Phi}_v = -N.$$

Note: We will also need these relationships later when we want to calculate an expression for $W_{p,S} = -D_p\tilde{G}$.

Proof: Since $\vec{\Phi}_u$ and $\vec{\Phi}_v$ are tangent vectors $\vec{N} \cdot \vec{\Phi}_u = 0$ and $\vec{N} \cdot \vec{\Phi}_v = 0$. Differentiating each equation by u and v, we get:

$$\vec{N}_u \cdot \vec{\Phi}_u + \vec{N} \cdot \vec{\Phi}_{uu} = 0 \qquad \vec{N}_u \cdot \vec{\Phi}_v + \vec{N} \cdot \vec{\Phi}_{vu} = 0$$

$$\vec{N}_u \cdot \vec{\Phi}_u = -\vec{N} \cdot \vec{\Phi}_{uu} = -L \qquad \vec{N}_u \cdot \vec{\Phi}_v = -\vec{N} \cdot \vec{\Phi}_{vu} = -M$$

$$\vec{N}_v \cdot \vec{\Phi}_u + \vec{N} \cdot \vec{\Phi}_{uv} = 0 \qquad \vec{N}_v \cdot \vec{\Phi}_v + \vec{N} \cdot \vec{\Phi}_{vv} = 0$$

$$\vec{N}_v \cdot \vec{\Phi}_u = -\vec{N} \cdot \vec{\Phi}_{uv} = -M \qquad \vec{N}_u \cdot \vec{\Phi}_v = -\vec{N} \cdot \vec{\Phi}_{vv} = -N.$$

Recall that for a vector $\vec{w} = a\vec{\Phi}_u + b\vec{\Phi}_v \in T_pS$, we defined

$$du(\vec{w}) = a \text{ and } dv(\vec{w}) = b.$$

To show that:

$$L\,du(\vec{w}_1)du(\vec{w}_2) + M\,du(\vec{w}_1)dv\,(\vec{w}_2) + M\,du(\vec{w}_2)dv(\vec{w}_1) + N\,dv(\vec{w}_1)dv(\vec{w}_2)$$

equals $< W_{p,s}(\vec{w}_1), \vec{w}_2 >$, we just need to show this for basis vectors $\vec{\Phi}_u$ and $\vec{\Phi}_v$ for T_pS.

Case 1: $\vec{w}_1 = \vec{w}_2 = \vec{\Phi}_u$

$$L\,du(\vec{\Phi}_u)du(\vec{\Phi}_u) + M\,du(\vec{\Phi}_u)dv(\vec{\Phi}_u) + M\,du(\vec{\Phi}_u)dv(\vec{\Phi}_u) + N dv(\vec{\Phi}_u)dv(\vec{\Phi}_u) = L$$

Since, $du(\vec{\Phi}_u) = 1,\ dv(\vec{\Phi}_u) = 0$

$$< W_{p,S}(\vec{\Phi}_u), \vec{\Phi}_u > =< -D_p\tilde{G}(\vec{\Phi}_u), \vec{\Phi}_u > = -< \vec{N}_u, \vec{\Phi}_u > = L.$$

Case 2: $\vec{w}_1 = \vec{\Phi}_u,\ \vec{w}_2 = \vec{\Phi}_v$

$$L\,du(\vec{\Phi}_u)du(\vec{\Phi}_v) + M\,du(\vec{\Phi}_u)dv(\vec{\Phi}_v) + M\,du(\vec{\Phi}_v)dv(\vec{\Phi}_u) + N dv(\vec{\Phi}_u)dv(\vec{\Phi}_v) = M$$

$$< W_{p,S}(\vec{\Phi}_u), \vec{\Phi}_v > =< -D_p\tilde{G}(\vec{\Phi}_u), \vec{\Phi}_v > = -< \vec{N}_u, \vec{\Phi}_v > = M.$$

Similarly, when $\vec{w}_1 = \vec{\Phi}_v$ and $\vec{w}_2 = \vec{\Phi}_u$.

Case 3: $\vec{w}_1 = \vec{\Phi}_v,\ \vec{w}_2 = \vec{\Phi}_v$

$$L\, du(\vec{\Phi}_v)du(\vec{\Phi}_v) + M\, du(\vec{\Phi}_v)dv(\vec{\Phi}_v) + M\, du(\vec{\Phi}_v)dv(\vec{\Phi}_v) + Ndv(\vec{\Phi}_v)dv(\vec{\Phi}_v) = N$$

$$< W_{p,S}(\vec{\Phi}_v), \vec{\Phi}_v > = < -D_p\tilde{G}(\vec{\Phi}_v), \vec{\Phi}_v > = < -\vec{N}_v, \vec{\Phi}_v > = N.$$

Ex. Calculate the Gauss map for the paraboloid $z = x^2 + y^2$. Find its image in S^2.

We can parametrize $z = x^2 + y^2$ by

$$\vec{\Phi}(u,v) = (u, v, u^2 + v^2)$$

$$\vec{\Phi}_u(u,v) = (1, 0, 2u)$$

$$\vec{\Phi}_v(u,v) = (0, 1, 2v)$$

$$\vec{\Phi}_u \times \vec{\Phi}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2u\vec{i} - 2v\vec{j} + \vec{k}$$

$$\left\|\vec{\Phi}_u \times \vec{\Phi}_v\right\| = \sqrt{4u^2 + 4v^2 + 1}$$

$$\vec{N}(u,v) = \frac{(-2u, -2v, 1)}{\sqrt{1+4u^2+4v^2}}$$

$$\tilde{G}(u, v, u^2 + v^2) = \frac{(-2u, -2v, 1)}{\sqrt{1+4u^2+4v^2}}.$$

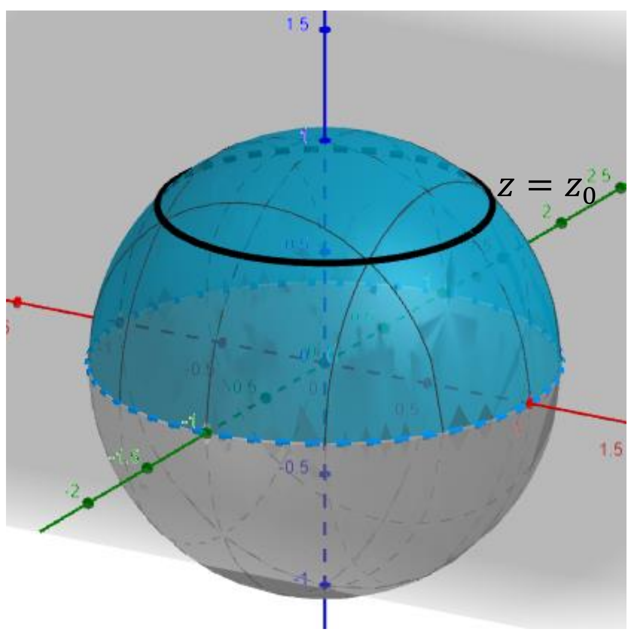
If we change to cylindrical coordinates:

$$\tilde{G}(r,\theta,z) = \frac{(-2r\cos\theta, -2r\sin\theta, 1)}{\sqrt{4r^2+1}}.$$

So $z = f(r) = \frac{1}{\sqrt{4r^2+1}}$ is 1-1 from $0 \le r < \infty$ onto $(0,1]$.

So for any $0 < z_0 \le 1$, there is a unique r_0 such that $\frac{1}{\sqrt{4{r_0}^2+1}} = z_0$.

For that r_0, $0 \le \theta \le 2\pi$ makes $\frac{(-2r_0\cos\theta, -2r_0\sin\theta, 1)}{\sqrt{4{r_0}^2+1}}$ a circle.



Thus, the image of the Gauss map is the upper hemisphere, not including the equator in the x-y plane.

Ex. Calculate the Gauss map for the cylinder in $\mathbb{R}^3$ given by $x^2 + y^2 = 1$.

What is the image of the Gauss map in S^2?

$$\vec{\Phi}(u, v) = (\cos u, \sin u, v) \;;\; 0 \le u \le 2\pi, v \in \mathbb{R}$$

$$\vec{\Phi}_u = (-\sin u, \cos u, 0)$$

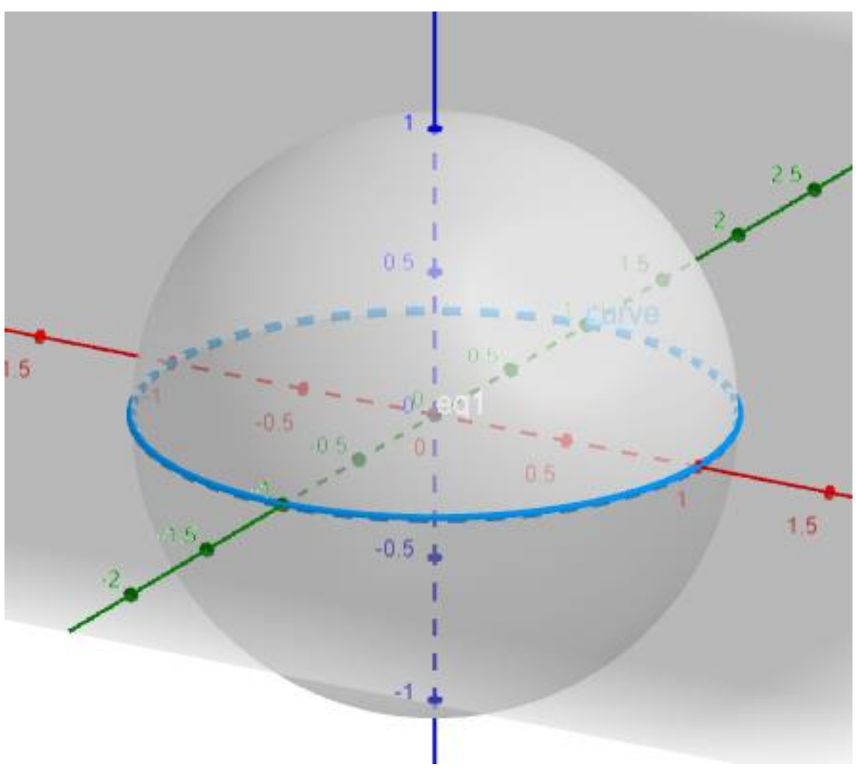
$$\vec{\Phi}_v = (0, 0, 1)$$

$$\vec{\Phi}_u \times \vec{\Phi}_v = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos u)\,\vec{\imath} + (\sin u)\,\vec{\jmath}\,.$$

This is already a unit vector, so we can write:

$$\tilde{G}(\cos u, \sin u, v) = (\cos u, \sin u, 0)\,; \quad 0 \le u \le 2\pi.$$

Thus the image of the Gauss map is the circle $x^2 + y^2 = 1, \; z = 0$.
Notice that in this example, the image of the Gauss map is not a regular surface.



Ex. Find the image of the Gauss map for $z = \sqrt{1 + x^2 + y^2}$ (the upper half of a 2 sheeted hyperboloid).

We can parametrize $z = \sqrt{1 + x^2 + y^2}$ by:

$$\vec{\Phi}(u, v) = \left(u, v, (1 + u^2 + v^2)^{\frac{1}{2}}\right).$$

Then we have:

$$\vec{\Phi}_u(u, v) = \left(1, 0, \frac{u}{(1+u^2+v^2)^{\frac{1}{2}}}\right) \qquad \vec{\Phi}_v(u, v) = \left(0, 1, \frac{v}{(1+u^2+v^2)^{\frac{1}{2}}}\right)$$

$$\vec{\Phi}_u \times \vec{\Phi}_v = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 0 & \frac{u}{(1+u^2+v^2)^{\frac{1}{2}}} \\ 0 & 1 & \frac{v}{(1+u^2+v^2)^{\frac{1}{2}}} \end{vmatrix} = \frac{-u}{(1+u^2+v^2)^{\frac{1}{2}}}\vec{\imath} - \frac{v}{(1+u^2+v^2)^{\frac{1}{2}}}\vec{\jmath} + \vec{k}$$

$$\vec{N}(u, v) = \frac{\vec{\Phi}_u \times \vec{\Phi}_v}{\|\vec{\Phi}_u \times \vec{\Phi}_v\|}$$

$$= \frac{< \frac{-u}{(1+u^2+v^2)^{\frac{1}{2}}}, \ \frac{-v}{(1+u^2+v^2)^{\frac{1}{2}}}, \ 1 >}{\sqrt{\frac{u^2}{1+u^2+v^2} + \frac{v^2}{1+u^2+v^2} + 1}}$$

$$= < \frac{-u}{(1+2u^2+2v^2)^{\frac{1}{2}}}, \ \frac{-v}{(1+2u^2+2v^2)^{\frac{1}{2}}}, \ \frac{\sqrt{1+u^2+v^2}}{(1+2u^2+2v^2)^{\frac{1}{2}}} >$$

$$= \tilde{G}\left(u, v, (1 + u^2 + v^2)^{\frac{1}{2}}\right).$$

In cylindrical coordinates we have:

$$\tilde{G}\left(r,\theta,\sqrt{1+r^2}\right) = < \frac{-rcos\theta}{\sqrt{1+2r^2}}, \frac{-rsin\theta}{\sqrt{1+2r^2}}, \frac{\sqrt{1+r^2}}{\sqrt{1+2r^2}} >.$$

So $z = f(r) = \frac{\sqrt{1+r^2}}{\sqrt{1+2r^2}}$ by division we get $\frac{1+r^2}{1+2r^2} = \frac{1}{2}\left(1 + \frac{1}{1+2r^2}\right)$,

$$= \sqrt{\frac{1}{2}\left(1 + \frac{1}{1+2r^2}\right)}$$

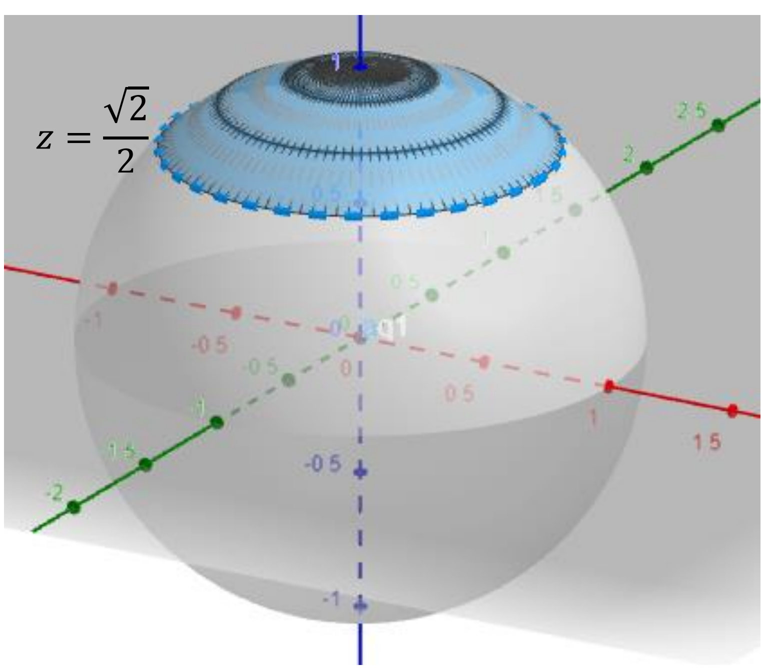
which is a strictly decreasing function of $r \geq 0$ $\quad (f'(r) < 0, \ \ r > 0)$.

$$\lim_{r\to\infty} \sqrt{\frac{1}{2}\left(1 + \frac{1}{1+2r^2}\right)} = \frac{\sqrt{2}}{2},$$

So $z = f(r)$ is a 1-1 map of $r \geq 0$ onto $\left(\frac{\sqrt{2}}{2}, 1\right]$.

And for each $\frac{\sqrt{2}}{2} < r_0 \leq 1$, $< \frac{-r_0 cos\theta}{\sqrt{1+2{r_0}^2}}, \frac{-r_0 sin\theta}{\sqrt{1+2{r_0}^2}}, \frac{\sqrt{1+{r_0}^2}}{\sqrt{1+2{r_0}^2}} >$ is a circle.

Thus the image of the Gauss map is the points in S^2 such that $\frac{\sqrt{2}}{2} < z \leq 1$.



Ex. Find the image of the Gauss map for the helicoid given by:

$$\vec{\Phi}(u,v) = (\mathrm{v}\cos u\,, v\sin u\,, u); \quad u \in \mathbb{R}, \quad -\sqrt{3} < v < \sqrt{3}.$$

$$\vec{\Phi}_u = (-v\sin u\,, v\cos u\,, 1) \qquad \vec{\Phi}_v = (cosu, sinu, 0)$$

$$\vec{\Phi}_u \times \vec{\Phi}_v = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -v\sin u & \mathrm{v}\cos u & 1 \\ cosu & sinu & 0 \end{vmatrix} = -(\sin u)\,\vec{\imath} + (\cos u)\,\vec{\jmath} - v\vec{k}\,.$$

$$\vec{N}(u,v) = \frac{\vec{\Phi}_u \times \vec{\Phi}_v}{\|\vec{\Phi}_u \times \vec{\Phi}_v\|} = \frac{<-sinu, cosu, -v>}{\sqrt{1+v^2}} = \tilde{G}(vcosu, vsinu, u).$$

$$z = f(v) = \frac{-v}{\sqrt{1+v^2}}; \quad -\sqrt{3} < v < \sqrt{3}.$$

$f'(v) = -\frac{1}{(1+v^2)^{\frac{3}{2}}} < 0 \implies f(v)$ is strictly decreasing for all v.

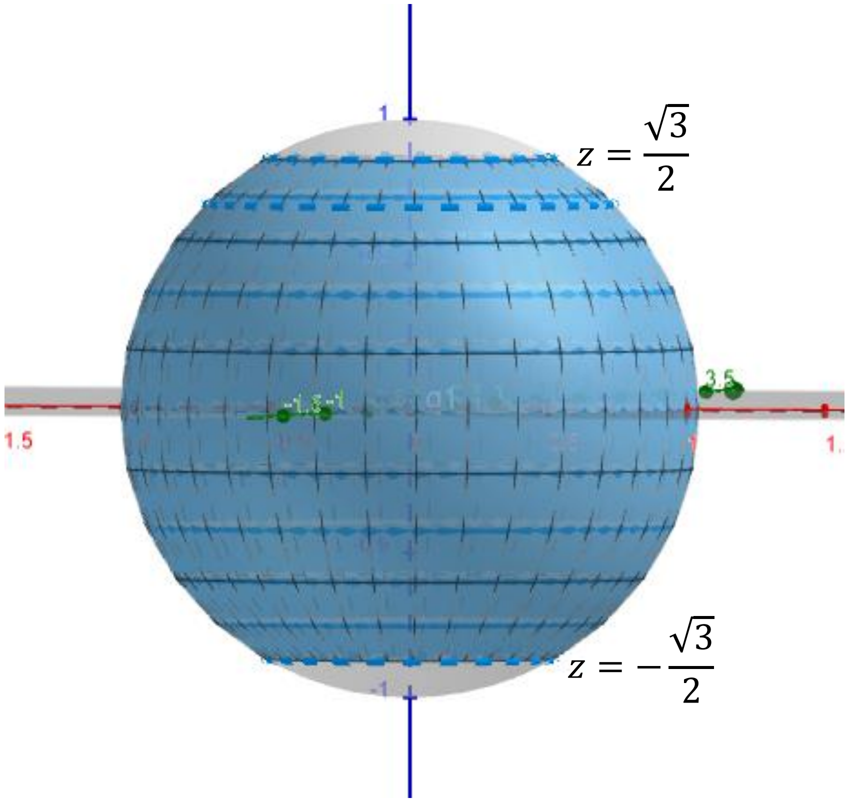
Since $f(-\sqrt{3}) = \frac{\sqrt{3}}{2}, \quad f(\sqrt{3}) = \frac{-\sqrt{3}}{2}$

$\implies \frac{-\sqrt{3}}{2} < z < \frac{\sqrt{3}}{2}$ when $-\sqrt{3} < v < \sqrt{3}$.

For any fixed $v_0, \quad -\sqrt{3} < v_0 < \sqrt{3}$,

$\tilde{G}(v_0 cosu, v_0 sinu, u) = \frac{<-sinu, cosu, -v_0>}{\sqrt{1+{v_0}^2}}$ is a circle.

$\implies$ Image of the Gauss map is the points in S^2 where $\frac{-\sqrt{3}}{2} < z < \frac{\sqrt{3}}{2}$.



Note: If we took v such that $-\infty < v < \infty$, then the image of the Gauss map would be S^2 minus the north and south poles.