## The Second Fundamental Form
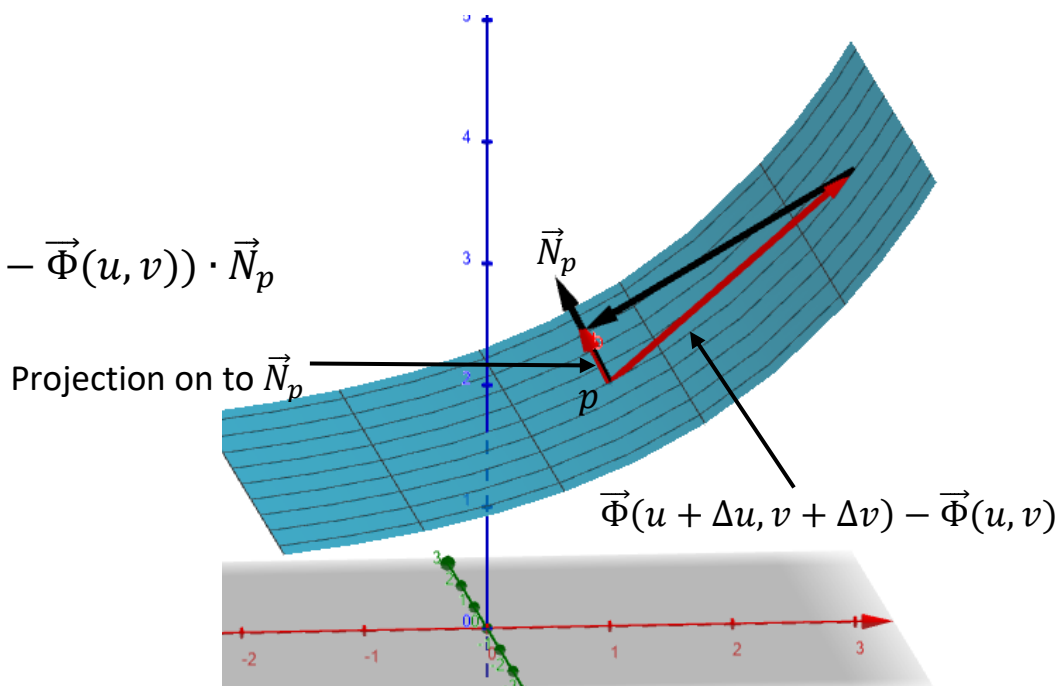
We saw earlier that a smooth curve in $\mathbb{R}^3$ is determined (up to a rigid transformation) by its curvature and torsion (in $\mathbb{R}^2$ it's determined by its signed curvature). For a smooth surface in $\mathbb{R}^3$, it's determined by its first and second fundamental forms. To define the curvature of a surface, we imitate the approach to defining the curvature of a curve.

Let $\vec{\Phi}: U \subseteq \mathbb{R}^2 \to S$ be a surface patch for the surface, $S$. Since $\vec{\Phi}_u \times \vec{\Phi}_v \neq \vec{0}$, $\vec{\Phi}_u \times \vec{\Phi}_v$ is a vector perpendicular to the tangent plane determined by $\vec{\Phi}_u, \vec{\Phi}_v$ ($\vec{\Phi}_u$ and $\vec{\Phi}_v$ are tangent vectors to $S$ at $\vec{\Phi}(u_0, v_0)$). Thus, $(\vec{\Phi}_u \times \vec{\Phi}_v)(u_0, v_0)$ is normal to the tangent plane, $T_p S$, where $p = \vec{\Phi}(u_0, v_0)$. So the vector: $\vec{N}_p = \dfrac{(\vec{\Phi}_u \times \vec{\Phi}_v)(u_0, v_0)}{\left\| \vec{\Phi}_u \times \vec{\Phi}_v (u_0, v_0) \right\|}$ is a unit normal vector to $T_p S$.

Suppose we start at a point, $p = \vec{\Phi}(u, v)$, on the surface $S$, and consider the vector $\vec{\Phi}(u + \Delta u, v + \Delta v) - \vec{\Phi}(u, v)$. If the length of the projection of this vector onto the unit normal vector $\vec{N}_p$ is "large" we should expect a "large" curvature. This length is the deviation of $\vec{\Phi}(u + \Delta u, v + \Delta v) - \vec{\Phi}(u, v)$ from the tangent plane $T_p S$.

Length of projection $=$

$$(\vec{\Phi}(u + \Delta u, v + \Delta v) - \vec{\Phi}(u, v)) \cdot \vec{N}_p$$

Projection on to $\vec{N}_p$

$\vec{N}_p$

$p$

$\vec{\Phi}(u + \Delta u, v + \Delta v) - \vec{\Phi}(u, v)$

Using Taylor's Theorem in two variables with remainder $R$ we get:

$$\vec{\Phi}(u + \Delta u, v + \Delta v) - \vec{\Phi}(u, v)$$

$$= \vec{\Phi}_u(\Delta u) + \vec{\Phi}_v(\Delta v) + \frac{1}{2}\left(\vec{\Phi}_{uu}(\Delta u)^2 + 2\vec{\Phi}_{uv}(\Delta u)(\Delta v) + \vec{\Phi}_{vv}(\Delta v)^2\right) + R$$

where $\dfrac{R}{(\Delta u)^2 + (\Delta v)^2}$ tends to zero as $(\Delta u)^2 + (\Delta v)^2$ tends to $0$.

Now $\vec{\Phi}_u$ and $\vec{\Phi}_v$ are tangent vectors to $S$ at $p$ and hence are perpendicular to $\vec{N}_p$. So we have:

$$\left(\vec{\Phi}(u + \Delta u, v + \Delta v) - \vec{\Phi}(u, v)\right) \cdot \vec{N}_p$$

$$= [(\vec{\Phi}_u(\Delta u) + \vec{\Phi}_v(\Delta v)$$

$$+ \frac{1}{2}\left(\vec{\Phi}_{uu}(\Delta u)^2 + 2\vec{\Phi}_{uv}(\Delta u)(\Delta v) + \vec{\Phi}_{vv}(\Delta v)^2 + R\right)] \cdot \vec{N}_p$$

$$= [\frac{1}{2}\left(\vec{\Phi}_{uu} \cdot \vec{N}_p(\Delta u)^2 + 2\vec{\Phi}_{uv} \cdot \vec{N}_p(\Delta u)(\Delta v) + \vec{\Phi}_{vv} \cdot \vec{N}_p(\Delta v)^2\right) + R] \cdot \vec{N}_p \; .$$

Define:     $L = \vec{\Phi}_{uu} \cdot \vec{N}_p$

$$M = \vec{\Phi}_{uv} \cdot \vec{N}_p$$

$$N = \vec{\Phi}_{vv} \cdot \vec{N}_p$$

$L(\Delta u)^2 + 2M(\Delta u)(\Delta v) + N(\Delta v)^2$ is the analogue for a surface of the term $\kappa(\Delta t)^2$ for a curve. One calls the expression:

$$L\,du^2 + 2M\,dudv + N\,dv^2$$

**the second fundamental form** of the surface patch $\vec{\Phi}$. Equivalently, we can think of the second fundamental form as:

$$(du \quad dv)\begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix} = L\,du^2 + 2M\,dudv + N\,dv^2.$$

Ex. Parametrize the plane $Ax + By + Cz = D$ and compute the second fundamental form for that parametrization.

Solving for $z$, we get:

$$z = -\frac{A}{C}x - \frac{B}{C}y + \frac{D}{C}.$$

So we can parametrize the surface by:

$$\vec{\Phi}(u, v) = \left(u, v, \ -\frac{A}{C}u - \frac{B}{C}v + \frac{D}{C}\right).$$

Notice that: $\vec{\Phi}_{uu} = 0$ ; $\vec{\Phi}_{uv} = 0, \vec{\Phi}_{vv} = 0.$

Thus, $L = M = N = 0$ and the second fundamental form of a plane is $0$.

Ex. Calculate the second fundamental form for a sphere of radius $R$

parametrized by:

$$\vec{\Phi}(\phi, \theta) = (R \cos \theta \sin \phi, R \sin \theta \sin \phi, R \cos \phi); \ 0 < \phi < \pi, 0 < \theta < 2\pi.$$

$$\vec{\Phi}_{\phi} = (R\cos \theta \cos \phi, R \sin \theta \cos \phi, -R\sin \phi)$$

$$\vec{\Phi}_{\theta} = (-R \sin \theta \sin \phi, R\cos \theta \sin \phi, 0)$$

$$\vec{\Phi}_{\phi} \times \vec{\Phi}_{\theta} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ R \cos \theta \cos \phi & R \sin \theta \cos \phi & -R \sin \phi \\ -R \sin \theta \sin \phi & R \cos \theta \sin \phi & 0 \end{vmatrix}$$

$$= (R^2 \sin^2 \phi \cos \theta)\vec{i} + (R^2 \sin^2 \phi \sin \theta)\vec{j} + (R^2 \sin \phi \cos \phi)\vec{k}.$$

$$\left\| \vec{\Phi}_\phi \times \vec{\Phi}_\theta \right\| = \sqrt{R^4 \sin^4 \phi \cos^2 \theta + R^4 \sin^4 \phi \sin^2 \theta + R^4 \sin^2 \phi \cos^2 \phi}$$

$$= R^2 \sin \phi$$

So:

$$\vec{N} = \frac{\vec{\Phi}_\phi \times \vec{\Phi}_\theta}{\left\| \vec{\Phi}_\phi \times \vec{\Phi}_\theta \right\|} = (\sin \phi \cos \theta)\vec{\imath} + (\sin \phi \sin \theta)\vec{\jmath} + (\cos \phi)\vec{k}.$$

$$\vec{\Phi}_{\phi\phi} = (-R \cos \theta \sin \phi)\vec{\imath} - (R \sin \theta \sin \phi)\vec{\jmath} - (R \cos \phi)\vec{k}$$

$$\vec{\Phi}_{\phi\theta} = (-R \sin \theta \cos \phi)\vec{\imath} + (R \cos \theta \cos \phi)\vec{\jmath}$$

$$\vec{\Phi}_{\theta\theta} = (-R \cos \theta \sin \phi)\vec{\imath} - (R \sin \theta \sin \phi)\vec{\jmath}.$$

$$L = \vec{\Phi}_{\phi\phi} \cdot \vec{N} = -R \cos^2 \theta \sin^2 \phi - R \sin^2 \theta \sin^2 \phi - R \cos^2 \phi = -R$$

$$M = \vec{\Phi}_{\phi\theta} \cdot \vec{N} = -R \sin \theta \cos \theta \sin \phi \cos \phi + R \sin \theta \cos \theta \sin \phi \cos \phi = 0$$

$$N = \vec{\Phi}_{\theta\theta} \cdot \vec{N} = -R \cos^2 \theta \sin^2 \phi - R \sin^2 \theta \sin^2 \phi = -R \sin^2 \phi$$

Thus, the second fundamental form is:

$$-R \, d\phi^2 - R \sin^2 \phi \, d\theta^2.$$

Equivalently we can write:

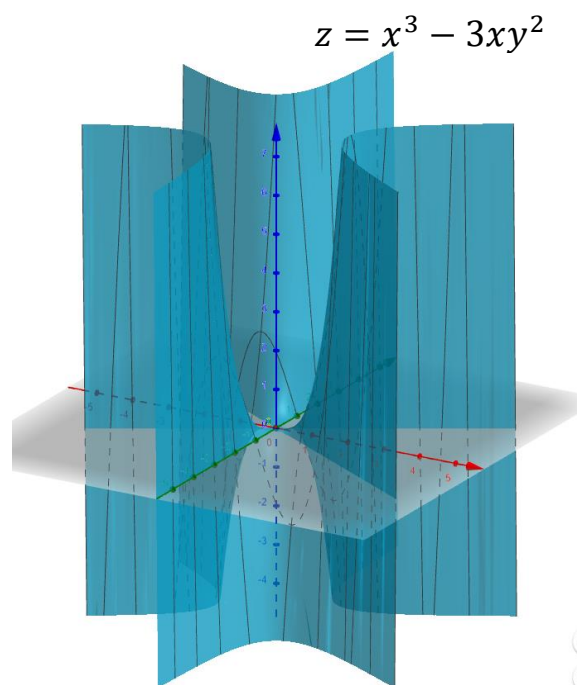$$\begin{pmatrix} -R & 0 \\ 0 & -R \sin^2 \phi \end{pmatrix}.$$

Ex.  Find the second fundamental form for the monkey saddle, $z = x^3 - 3xy^2$,

parametrized by:

$$\vec{\Phi}(u, v) = (u, v, u^3 - 3uv^2).$$

$z = x^3 - 3xy^2$

Then we have:  $\vec{\Phi}_u = (1, 0, 3u^2 - 3v^2)$

$\vec{\Phi}_v = (0, 1, -6uv)$

$$\vec{\Phi}_u \times \vec{\Phi}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 3u^2 - 3v^2 \\ 0 & 1 & -6uv \end{vmatrix}$$

$$= (3v^2 - 3u^2)\vec{i} + (6uv)\vec{j} + \vec{k}$$

$$\vec{N} = \frac{\vec{\Phi}_u \times \vec{\Phi}_v}{\|\vec{\Phi}_u \times \vec{\Phi}_v\|} = \frac{(3v^2-3u^2, 6uv, 1)}{\sqrt{(3v^2-3u^2)^2+(6uv)^2+1}} = \frac{(3v^2-3u^2, 6uv, 1)}{\sqrt{1+(3v^2+3u^2)^2}}$$

$\vec{\Phi}_{uu} = (0, 0, 6u)$

$\vec{\Phi}_{uv} = (0, 0, -6v)$

$\vec{\Phi}_{vv} = (0, 0, -6u)$

$$L = \vec{\Phi}_{uu} \cdot \vec{N} = \frac{6u}{\sqrt{1+(3v^2+3u^2)^2}}$$

$$M = \vec{\Phi}_{uv} \cdot \vec{N} = \frac{-6v}{\sqrt{1+(3v^2+3u^2)^2}}$$

$$N = \vec{\Phi}_{vv} \cdot \vec{N} = \frac{-6u}{\sqrt{1+(3v^2+3u^2)^2}}$$

So the second fundamental form is:

$$\frac{1}{\sqrt{1+(3v^2+3u^2)^2}}[6u\,du^2 - 12v\,dudv - 6u\,dv^2]$$

Equivalently:

$$\frac{1}{\sqrt{1+(3v^2+3u^2)^2}}\begin{pmatrix} 6u & -6v \\ -6v & -6u \end{pmatrix}.$$

Notice that at $u = 0, v = 0$, the second fundamental form is zero.

Def.  Let $S$ be a regular (**orientable**, i.e., any two surface patches give the same unit Normal) smooth surface. Let the second fundamental form be:

$$H = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

A point, $p \in S$, is called:

1) **Elliptic** if $\det(H) > 0$

2) **Hyperbolic** if $\det(H) < 0$

3) **Parabolic** if $\det(H) = 0$, but not all $L(p), M(p), N(p)$ are 0

4) **Planar** if $L(p) = M(p) = N(p) = 0$.

So the point $(0,0,0)$ on the monkey saddle (corresponding to $u = 0, v = 0$) is a planar point. All other points on the monkey saddle are hyperbolic because:

$$\det(H) = \frac{1}{1+(3v^2+3u^2)^2} \det \begin{pmatrix} 6u & -6v \\ -6v & -6u \end{pmatrix}$$

$$= \frac{1}{1+(3v^2+3u^2)^2} (-36)(u^2 + v^2) < 0 \quad \text{(if } u \text{ or } v \neq 0\text{)}.$$

Ex.   Find which points of the torus below are elliptical, hyperbolic, parabolic, or planar.

$$\vec{\Phi}(u,v) = ((2 + \cos v)\cos u, (2 + \cos v)\sin u, \sin v)$$

$$0 < u < 2\pi ; \quad -\frac{\pi}{2} < v < \frac{3\pi}{2}.$$

If one calculates: $\vec{\Phi}_u, \ \vec{\Phi}_v, \ \vec{\Phi}_u \times \vec{\Phi}_v, \vec{N} = \frac{\vec{\Phi}_u \times \vec{\Phi}_v}{\|\vec{\Phi}_u \times \vec{\Phi}_v\|}, \vec{\Phi}_{uu}, \vec{\Phi}_{uv},$ and $\vec{\Phi}_{vv}$ one finds:

$$\vec{N} = (-\cos u \cos v, -\sin u \cos v, -\sin v)$$

$$\vec{\Phi}_{uu} = (-(2 + \cos v)\cos u, -(2 + \cos v)\sin u, 0)$$

$$\vec{\Phi}_{uv} = ((\sin u)\sin v, -(\cos u)\sin v, 0)$$

$$\vec{\Phi}_{vv} = (-(\cos v)\cos u, -(\cos v)\sin u, -\sin v).$$

$$L = \vec{\Phi}_{uu} \cdot \vec{N} = (2 + \cos v) \cos v$$

$$M = \vec{\Phi}_{uv} \cdot \vec{N} = 0$$

$$N = \vec{\Phi}_{vv} \cdot \vec{N} = 1.$$

So the second fundamental form is:

$$H = \begin{pmatrix} (2 + \cos v) \cos v & 0 \\ 0 & 1 \end{pmatrix}$$

and $\quad \det H = (2 + \cos v) \cos v.$

Since $-1 \le \cos v \le 1$ we have, $1 \le 2 + \cos v \le 3$. So the sign of $\det H$ is the same as $\cos v$.

$\det H = 0 \Rightarrow \cos v = 0 \Rightarrow v = \dfrac{\pi}{2}$ or $-\dfrac{\pi}{2} \quad \Longrightarrow$ Parabolic points

$\det H > 0$ when $\cos v > 0 \Rightarrow -\dfrac{\pi}{2} < v < \dfrac{\pi}{2} \quad \Longrightarrow$ Elliptic points

$\det H < 0$ when $\cos v < 0 \Rightarrow \dfrac{\pi}{2} < v < \dfrac{3\pi}{2} \quad \Longrightarrow$ Hyperbolic points