The Second Fundamental Form

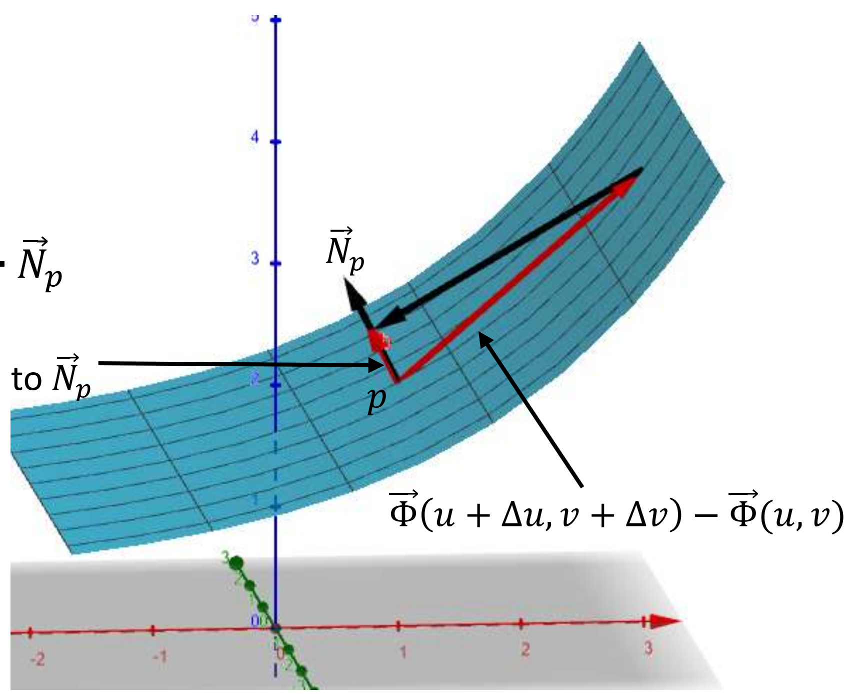
We saw earlier that a smooth curve in $\mathbb{R}^3$ is determined (up to a rigid transformation) by its curvature and torsion (in $\mathbb{R}^2$ it's determined by its signed curvature). For a smooth surface in $\mathbb{R}^3$, it's determined by its first and second fundamental forms. To define the curvature of a surface, we imitate the approach to defining the curvature of a curve.

Let $\vec{\Phi}: U \subseteq \mathbb{R}^2 \to S$ be a surface patch for the surface, S. Since $\vec{\Phi}_u \times \vec{\Phi}_v \neq \vec{0}$, $\vec{\Phi}_u \times \vec{\Phi}_v$ is a vector perpendicular to the tangent plane determined by $\vec{\Phi}_u, \vec{\Phi}_v$ ($\vec{\Phi}_u$ and $\vec{\Phi}_v$ are tangent vectors to S at $\vec{\Phi}(u_0, v_0)$). Thus, $\left(\vec{\Phi}_u \times \vec{\Phi}_v\right)(u_0, v_0)$ is normal to the tangent plane, T_pS, where $p = \vec{\Phi}(u_0, v_0)$. So the vector: $\vec{N}_p = \frac{\left(\vec{\Phi}_u \times \vec{\Phi}_v\right)(u_0,v_0)}{\left\|\vec{\Phi}_u \times \vec{\Phi}_v(u_0,v_0)\right\|}$ is a unit normal vector to T_pS.

Suppose we start at a point, $p = \vec{\Phi}(u, v)$, on the surface S, and consider the vector $\vec{\Phi}(u + \Delta u, v + \Delta v) - \vec{\Phi}(u, v)$. If the length of the projection of this vector onto the unit normal vector $\vec{N}_p$ is "large" we should expect a "large" curvature. This length is the deviation of $\vec{\Phi}(u + \Delta u, v + \Delta v) - \vec{\Phi}(u, v)$ from the tangent plane T_pS.

Length of projection $=$

$(\vec{\Phi}(u + \Delta u, v + \Delta v) - \vec{\Phi}(u, v)) \cdot \vec{N}_p$



Using Taylor's Theorem in two variables with remainder R we get:

$$\vec{\Phi}(u+\Delta u, v+\Delta v) - \vec{\Phi}(u,v)$$

$$= \vec{\Phi}_u(\Delta u) + \vec{\Phi}_v(\Delta v) + \tfrac{1}{2}\left(\vec{\Phi}_{uu}(\Delta u)^2 + 2\vec{\Phi}_{uv}(\Delta u)(\Delta v) + \vec{\Phi}_{vv}(\Delta v)^2\right) + R$$

where $\frac{\mathrm{R}}{(\Delta u)^2+(\Delta v)^2}$ tends to zero as $(\Delta u)^2 + (\Delta v)^2$ tends to 0.

Now $\vec{\Phi}_u$ and $\vec{\Phi}_v$ are tangent vectors to S at p and hence are perpendicular to $\vec{N}_p$. So we have:

$$\left(\vec{\Phi}(u+\Delta u, v+\Delta v) - \vec{\Phi}(u,v)\right) \cdot \vec{N}_p$$

$$= [(\vec{\Phi}_u(\Delta u) + \vec{\Phi}_v(\Delta\, v) + \frac{1}{2}\left(\vec{\Phi}_{uu}(\Delta u)^2 + 2\vec{\Phi}_{uv}(\Delta u)(\Delta v) + \vec{\Phi}_{vv}(\Delta v)^2 + \mathrm{R}\right)] \cdot \vec{N}_p$$

$$= [\tfrac{1}{2}\left(\vec{\Phi}_{uu}\cdot\vec{N}_p(\Delta u)^2 + 2\vec{\Phi}_{uv}\cdot\vec{N}_p(\Delta u)(\Delta v) + \vec{\Phi}_{vv}\cdot\vec{N}_p(\Delta v)^2\right) + \mathrm{R}]\cdot\vec{N}_p\ .$$

Define:

$$L = \vec{\Phi}_{uu}\cdot\vec{N}_p$$

$$M = \vec{\Phi}_{uv}\cdot\vec{N}_p$$

$$N = \vec{\Phi}_{vv}\cdot\vec{N}_p$$

$L(\Delta u)^2 + 2M(\Delta u)(\Delta v) + N(\Delta v)^2$ is the analogue for a surface of the term $\kappa(\Delta t)^2$ for a curve. One calls the expression:

$$L\ du^2 + 2M\ dudv + N\ dv^2$$

the second fundamental form of the surface patch $\vec{\Phi}$. Equivalently, we can think of the second fundamental form as:

$$(du \quad dv)\begin{pmatrix} L & M \\ M & N\end{pmatrix}\begin{pmatrix} du \\ dv\end{pmatrix} = L\ du^2 + 2M\ dudv + N\ dv^2.$$

Ex. Parametrize the plane $Ax + By + Cz = D$ and compute the second fundamental form for that parametrization.

Solving for z, we get:

$$z = -\frac{A}{C}x - \frac{B}{C}y + \frac{D}{C}.$$

So we can parametrize the surface by:

$$\vec{\Phi}(u, v) = \left(u, v,\ -\frac{A}{C}u - \frac{B}{C}v + \frac{D}{C}\right).$$

Notice that: $\vec{\Phi}_{uu} = 0$; $\vec{\Phi}_{uv} = 0, \vec{\Phi}_{vv} = 0$.

Thus, $L = M = N = 0$ and the second fundamental form of a plane is 0.

Ex. Calculate the second fundamental form for a sphere of radius R

parametrized by:

$$\vec{\Phi}(\phi, \theta) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi);\ \ 0 \le \phi \le \pi, 0 \le \theta \le 2\pi.$$

$$\vec{\Phi}_\phi = (\mathrm{R}\cos\theta\cos\phi, R\sin\theta\cos\phi, -\mathrm{R}\sin\phi)$$

$$\vec{\Phi}_\theta = (-R\sin\theta\sin\phi, \mathrm{R}\cos\theta\sin\phi, 0)$$

$$\vec{\Phi}_\phi \times \vec{\Phi}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ R\cos\theta\cos\phi & R\sin\theta\cos\phi & -R\sin\phi \\ -R\sin\theta\sin\phi & R\cos\theta\sin\phi & 0 \end{vmatrix}$$

$$= (R^2\sin^2\phi\cos\theta)\vec{i} + (R^2\sin^2\phi\sin\theta)\vec{j} + (R^2\sin\phi\cos\phi)\vec{k}.$$

$$\left\|\vec{\Phi}_\phi \times \vec{\Phi}_\theta\right\| = \sqrt{R^4 \sin^4\phi \cos^2\theta + R^4 \sin^4\phi \sin^2\theta + R^4 \sin^2\phi \cos^2\phi}$$

$$= R^2 \sin\phi$$

So:

$$\vec{N} = \frac{\vec{\Phi}_\phi \times \vec{\Phi}_\theta}{\left\|\vec{\Phi}_\phi \times \vec{\Phi}_\theta\right\|} = (\sin\phi\cos\theta)\vec{\imath} + (\sin\phi\sin\theta)\vec{\jmath} + (\cos\phi)\vec{k}.$$

$$\vec{\Phi}_{\phi\phi} = (-R\cos\theta\sin\phi)\vec{\imath} - (R\sin\theta\sin\phi)\vec{\jmath} - (R\cos\phi)\vec{k}$$

$$\vec{\Phi}_{\phi\theta} = (-R\sin\theta\cos\phi)\vec{\imath} + (R\cos\theta\cos\phi)\vec{\jmath}$$

$$\vec{\Phi}_{\theta\theta} = (-R\cos\theta\sin\phi)\vec{\imath} - (R\sin\theta\sin\phi)\vec{\jmath}.$$

$$L = \vec{\Phi}_{\phi\phi} \cdot \vec{N} = -R\cos^2\theta\sin^2\phi - R\sin^2\theta\sin^2\phi - R\cos^2\phi = -R$$

$$M = \vec{\Phi}_{\phi\theta} \cdot \vec{N} = -R\sin\theta\cos\theta\sin\phi\cos\phi + R\sin\theta\cos\theta\sin\phi\cos\phi = 0$$

$$N = \vec{\Phi}_{\theta\theta} \cdot \vec{N} = -R\cos^2\theta\sin^2\phi - R\sin^2\theta\sin^2\phi = -R\sin^2\phi$$

Thus, the second fundamental form is:

$$-R\ d\phi^2 - R\sin^2\phi\ \ d\theta^2.$$

Equivalently we can write:

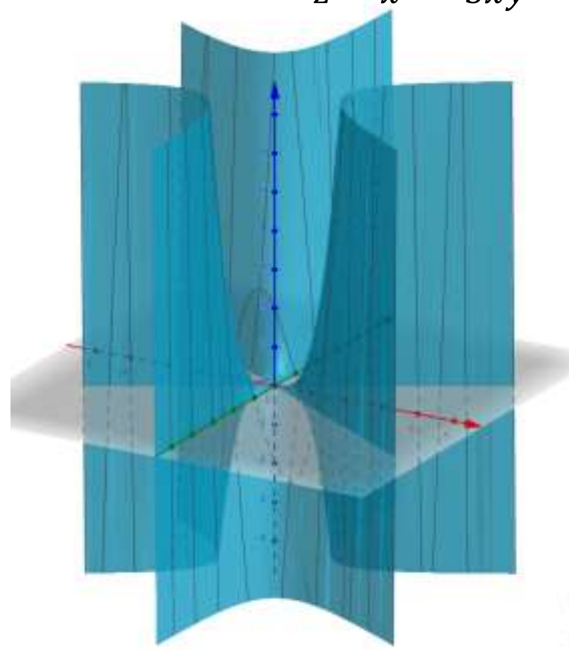
$$\begin{pmatrix} -R & 0 \\ 0 & -R\sin^2\phi \end{pmatrix}.$$

Ex. Find the second fundamental form for the monkey saddle, $z = x^3 - 3xy^2$,

parametrized by:

$$\vec{\Phi}(u, v) = (u, v, u^3 - 3uv^2).$$

$z = x^3 - 3xy^2$



Then we have: $\vec{\Phi}_u = (1, 0, 3u^2 - 3v^2)$

$$\vec{\Phi}_v = (0, 1, -6uv)$$

$$\vec{\Phi}_u \times \vec{\Phi}_v = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 0 & 3u^2 - 3v^2 \\ 0 & 1 & -6uv \end{vmatrix}$$

$$= (3v^2 - 3u^2)\vec{\imath} + (6uv)\vec{\jmath} + \vec{k}$$

$$\vec{N} = \frac{\vec{\Phi}_u \times \vec{\Phi}_v}{\|\vec{\Phi}_u \times \vec{\Phi}_v\|} = \frac{(3v^2 - 3u^2, 6uv, 1)}{\sqrt{(3v^2 - 3u^2)^2 + (6uv)^2 + 1}} = \frac{(3v^2 - 3u^2, 6uv, 1)}{\sqrt{1 + (3v^2 + 3u^2)^2}}$$

$$\vec{\Phi}_{uu} = (0, 0, 6u)$$

$$\vec{\Phi}_{uv} = (0, 0, -6v)$$

$$\vec{\Phi}_{vv} = (0, 0, -6u)$$

$$L = \vec{\Phi}_{uu} \cdot \vec{N} = \frac{6u}{\sqrt{1 + (3v^2 + 3u^2)^2}}$$

$$M = \vec{\Phi}_{uv} \cdot \vec{N} = \frac{-6v}{\sqrt{1 + (3v^2 + 3u^2)^2}}$$

$$N = \vec{\Phi}_{vv} \cdot \vec{N} = \frac{-6u}{\sqrt{1 + (3v^2 + 3u^2)^2}}$$

So the second fundamental form is:

$$\frac{1}{\sqrt{1+(3v^2+3u^2)^2}}\left[6u\,du^2 - 12v\,dudv - 6u\,dv^2\right]$$

Equivalently:

$$\frac{1}{\sqrt{1+(3v^2+3u^2)^2}}\begin{pmatrix} 6u & -6v \\ -6v & -6u \end{pmatrix}.$$

Notice that at $u = 0, v = 0$, the second fundamental form is zero.

Def. Let S be a regular (**orientable**, i.e., any two surface patches give the same unit Normal) smooth surface. Let the second fundamental form be:

$$H = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

A point, $p \in S$, is called:

1) **Elliptic** if $\det(H) > 0$

2) **Hyperbolic** if $\det(H) < 0$

3) **Parabolic** if $\det(H) = 0$, but not all $L(p), M(p), N(p)$ are 0

4) **Planar** if $L(p) = M(p) = N(p) = 0$.

So the point $(0,0,0)$ on the monkey saddle (corresponding to $u=0, v=0$) is a planar point. All other points on the monkey saddle are hyperbolic because:

$$\det(H) = \frac{1}{1+(3v^2+3u^2)^2}\det\begin{pmatrix} 6u & -6v \\ -6v & -6u \end{pmatrix}$$

$$= \frac{1}{1+(3v^2+3u^2)^2}(-36)(u^2+v^2) < 0 \quad \text{(if } u \text{ or } v \neq 0\text{)}.$$

Ex. Find which points of the torus below are elliptical, hyperbolic, parabolic, or planar.

$$\vec{\Phi}(u,v) = ((2+\cos v)\cos u\,, (2+\cos v)\sin u\,, \sin v)$$

$$0 \leq u \leq 2\pi\,;\ -\frac{\pi}{2} \leq v \leq \frac{3\pi}{2}\,.$$

If one calculates: $\vec{\Phi}_u,\ \vec{\Phi}_v,\ \vec{\Phi}_u \times \vec{\Phi}_v, \vec{N} = \frac{\vec{\Phi}_u \times \vec{\Phi}_v}{\|\vec{\Phi}_u \times \vec{\Phi}_v\|}, \vec{\Phi}_{uu}, \vec{\Phi}_{uv}$, and $\vec{\Phi}_{vv}$ one finds:

$$\vec{N} = (-\cos u \cos v\,, -\sin u \cos v\,, -\sin v)$$

$$\vec{\Phi}_{uu} = (-(2+\cos v)\cos u\,, -(2+\cos v)\sin u\,, 0)$$

$$\vec{\Phi}_{uv} = ((\sin u)\sin v\,, -(\cos u)\sin v\,, 0)$$

$$\vec{\Phi}_{vv} = (-(\cos v)\cos u\,, -(\cos v)\sin u\,, -\sin v).$$

$$L = \vec{\Phi}_{uu} \cdot \vec{N} = (2 + \cos v) \cos v$$

$$M = \vec{\Phi}_{uv} \cdot \vec{N} = 0$$

$$N = \vec{\Phi}_{vv} \cdot \vec{N} = 1.$$

So the second fundamental form is:

$$H = \begin{pmatrix} (2 + \cos v) \cos v & 0 \\ 0 & 1 \end{pmatrix}$$

and $\det H = (2 + \cos v) \cos v.$

Since $-1 \leq \cos v \leq 1$ we have, $1 \leq 2 + \cos v \leq 3$. So the sign of $\det H$ is the same as $\cos v$.

$\det H = 0 \Rightarrow \cos v = 0 \Rightarrow v = \frac{\pi}{2}$ or $-\frac{\pi}{2}$ $\Longrightarrow$ Parabolic points

$\det H > 0$ when $\cos v > 0 \Rightarrow -\frac{\pi}{2} < v < \frac{\pi}{2}$ $\Longrightarrow$ Elliptic points

$\det H < 0$ when $\cos v < 0 \Rightarrow \frac{\pi}{2} < v < \frac{3\pi}{2}$ $\Longrightarrow$ Hyperbolic points