The First Fundamental Form: Surface Area

One learns in vector calculus that the surface area of a region, R, on a surface, S, where $\vec{\Phi}: Q \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$, and $\vec{\Phi}(Q) = R$ is given by:

$$\text{Area}(R) = \iint_R \; dA = \iint_Q \left\|\vec{\Phi}_u \times \vec{\Phi}_v\right\| dudv.$$

Proposition: Let $\vec{\Phi}: Q \subseteq \mathbb{R}^2 \to S$ be a parametrization of a region $R \subseteq S$, then:

$$\text{Area}(R) = \iint_Q \sqrt{\det(g)} \; dudv = \iint_Q \sqrt{EG - F^2}\, dudv$$

where g is the metric tensor associated with $\vec{\Phi}$.

Proof: $\text{Area}(R) = \iint_Q \left\|\vec{\Phi}_u \times \vec{\Phi}_v\right\| dudv.$

$$\vec{\Phi}_u \times \vec{\Phi}_v = \begin{pmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix}$$

$$\vec{\Phi}_u \times \vec{\Phi}_v = \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)\vec{\imath} - \left(\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial z}{\partial u}\right)\vec{\jmath} + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\vec{k}$$

$$\left\|\vec{\Phi}_u \times \vec{\Phi}_v\right\| = \sqrt{\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)^2}.$$

$$E = \vec{\Phi}_u \cdot \vec{\Phi}_u = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2$$

$$G = \vec{\Phi}_v \cdot \vec{\Phi}_v = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2$$

$$F = \vec{\Phi}_u \cdot \vec{\Phi}_v = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}.$$

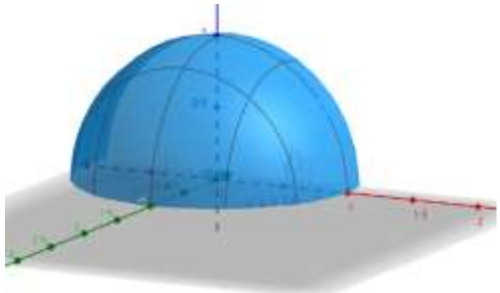
By a direct and unpleasant calculation, we get:

$$\|\vec{\Phi}_u \times \vec{\Phi}_v\| = \sqrt{(\vec{\Phi}_u \cdot \vec{\Phi}_u)(\vec{\Phi}_v \cdot \vec{\Phi}_v) - (\vec{\Phi}_u \cdot \vec{\Phi}_v)^2}$$

$$= \sqrt{EG - F^2} = \sqrt{\det g}\ .$$

Ex. Find the area of the upper unit hemisphere, H, using the parametrization: $H = \vec{\Phi}(U)$, where:

$$\vec{\Phi}(u, v) = (\cos v \sin u\,, \sin v \sin u\,, \cos u)\ ;\ \ 0 \leq u \leq \frac{\pi}{2},\ \ 0 \leq v \leq 2\pi.$$



We saw earlier that the metric tensor was: $g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 u \end{pmatrix}$.

$$A = \int_{v=0}^{2\pi} \int_{u=0}^{\frac{\pi}{2}} \sqrt{\sin^2 u}\ \ dudv = \int_{v=0}^{2\pi} \int_{u=0}^{\frac{\pi}{2}} \sin u\ \ dudv$$

($\sin u \geq 0$ for $0 \leq u \leq \frac{\pi}{2}$ so $|\sin u| = \sin u$)

$$A = \int_{v=0}^{2\pi} (-\cos u)\Big|_{u=0}^{u=\frac{\pi}{2}}\ dv = \int_{v=0}^{2\pi} 1\, dv = 2\pi.$$

Note: If we had used the parametrization,

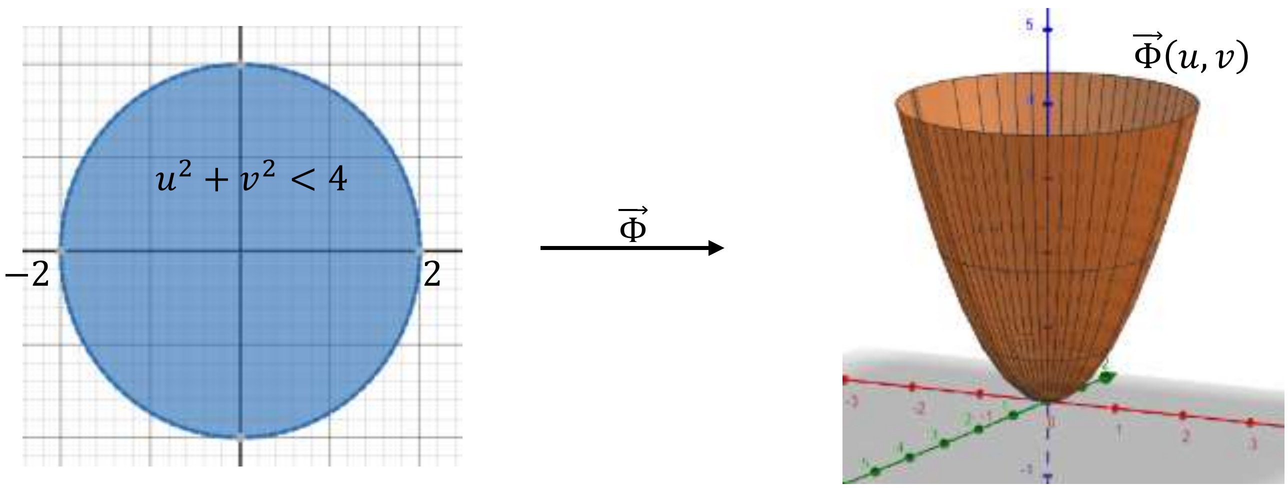
$$\vec{\Psi}(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right); \quad u^2+v^2 \le 1,$$

we would have gotten the same answer, but the calculation is a little messier.

Ex. Find the area of the paraboloid $z = x^2 + y^2$ below the plane $z = 4$.

We can parametrize the portion of the paraboloid by:

$$\vec{\Phi}(u,v) = (u, v, u^2+v^2); \quad u^2+v^2 < 4.$$



$$\vec{\Phi}_u = (1, 0, 2u), \qquad \vec{\Phi}_v = (0, 1, 2v):$$

$$E = \vec{\Phi}_u \cdot \vec{\Phi}_u = (1,0,2u)\cdot(1,0,2u) = 1+4u^2$$

$$F = \vec{\Phi}_u \cdot \vec{\Phi}_v = (1,0,2u)\cdot(0,1,2v) = 4uv$$
$$G = \vec{\Phi}_v \cdot \vec{\Phi}_v = (0,1,2v)\cdot(0,1,2v) = 1+4v^2$$

So the first fundamental form is:

$$(1+4u^2)du^2 + 8uv\, dudv + (1+4v^2)dv^2.$$

Equivalently, the metric tensor is:

$$g = \begin{pmatrix} 1+4u^2 & 4uv \\ 4uv & 1+4v^2 \end{pmatrix}$$

$$\det g = (1+4u^2)(1+4v^2) - 16u^2v^2 = 1 + 4u^2 + 4v^2.$$

Area$= \iint_{u^2+v^2\leq 4} \sqrt{\det g}\, dudv$

$$= \iint_{u^2+v^2\leq 4} \sqrt{1+4u^2+4v^2}\ dudv.$$

Changing to polar coordinates we get:

$$u^2 + v^2 = r^2$$

$$du\, dv = r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \left(\sqrt{1+4r^2}\right) r\, drd\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\frac{2}{3}\right)\left(\frac{1}{8}\right)(1+4r^2)^{\frac{3}{2}}\Big|_{r=0}^{r=r} d\theta$$

$$= \int_0^{2\pi} \frac{\left(17^{\frac{3}{2}}-1\right)}{12} d\theta = \frac{\left(17^{\frac{3}{2}}-1\right)}{12} (\theta)\Big|_0^{2\pi}$$

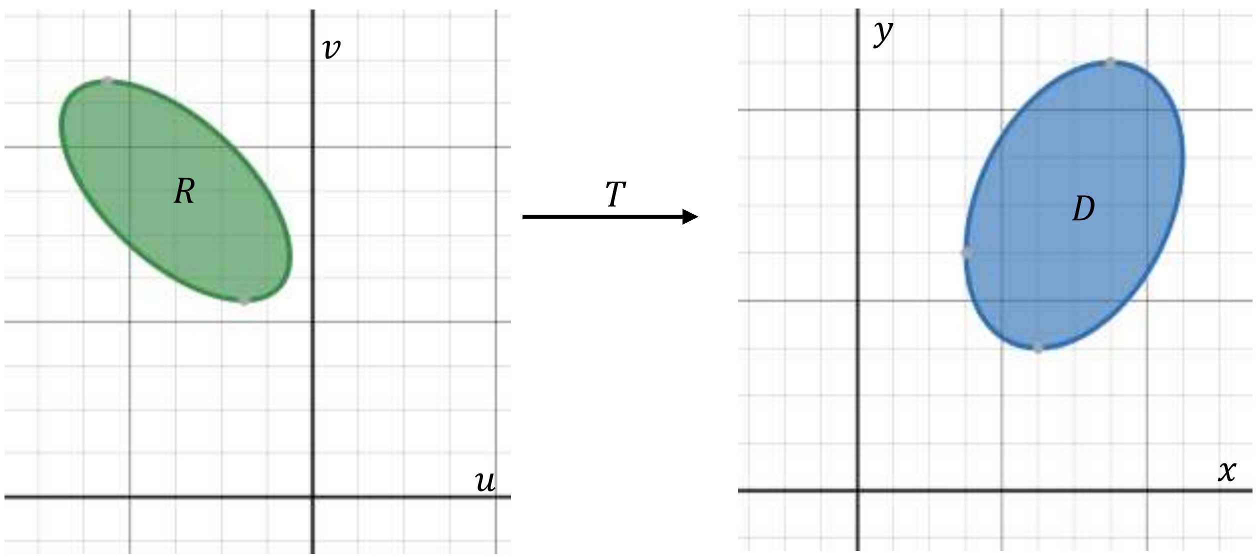
$$= \frac{\left(17^{\frac{3}{2}}-1\right)}{12}(2\pi) = \frac{\left(17^{\frac{3}{2}}-1\right)}{6}\pi\ .$$

Theorem (Change of Variables in $\mathbb{R}^2$): If $x = x(u, v),\ y = y(u, v)$ is a smooth change of variables on a region $D \subseteq \mathbb{R}^2$ then:

$$\iint_D f(x, y)dxdy = \iint_R f\big(x(u, v), y(u, v)\big)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv$$

where $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \neq 0$, and the map $T(u, v) = (x(u, v), y(u, v))$

maps R 1-1 (except perhaps on the boundary), smoothly onto D.



Proof: Let $\overrightarrow{\Phi}(u, v) = (x(u, v), y(u, v), 0)$.

By assumption, $\overrightarrow{\Phi}$ is a regular parametrization of D, a surface In $\mathbb{R}^2$.

$\overrightarrow{\Phi}_u = (x_u, y_u, 0)$, $\qquad$ $\overrightarrow{\Phi}_v = (x_v, y_v, 0)$:

$$E = \overrightarrow{\Phi}_u \cdot \overrightarrow{\Phi}_u = (x_u, y_u, 0) \cdot (x_u, y_u, 0) = x_u^2 + y_u^2$$

$$F = \overrightarrow{\Phi}_u \cdot \overrightarrow{\Phi}_v = (x_u, y_u, 0) \cdot (x_v, y_v, 0) = x_u x_v + y_u y_v$$

$$G = \overrightarrow{\Phi}_v \cdot \overrightarrow{\Phi}_v = (x_v, y_v, 0) \cdot (x_v, y_v, 0) = x_v^2 + y_v^2$$

So we have:

$$g = \begin{pmatrix} x_u^2 + y_u^2 & x_u x_v + y_u y_v \\ x_u x_v + y_u y_v & x_v^2 + y_v^2 \end{pmatrix}$$

$$\det(g) = (x_u^2 + y_u^2)(x_v^2 + y_v^2) - (x_u x_v + y_u y_v)^2$$
$$= (x_u y_v - x_v y_u)^2 = \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2$$

$$Area(D) = \iint_D \ dxdy = \iint_R \ \sqrt{\det(g)}dudv$$
$$= \iint_R \ |\frac{\partial(x,y)}{\partial(u,v)}|dudv.$$

Thus we have:

$$\iint_D \ f(x,y)dxdy = \iint_R \ f\big(x(u,v), y(u,v)\big)|\frac{\partial(x,y)}{\partial(u,v)}|dudv.$$