Level Surfaces and Quadric Surfaces

A level surface is a surface, S, given as:

$$S = \{(x, y, z) \in \mathbb{R}^3 |\ f(x, y, z) = 0\}$$

For example, the sphere of radius, R, centered at x_0, y_o, z_0 given by: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$ is a level surface since this is equivalent to:

$$f(x, y, z) = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 - R^2 = 0.$$

Theorem: Let S be a subset of $\mathbb{R}^3$ such that for each $p \in S$, there is an open set W of $\mathbb{R}^3$ containing p and a smooth map $f: W \to \mathbb{R}$ such that:

i) $S \cap W = \{(x, y, z) \in W |\ f(x, y, z) = 0\}$
ii) The gradient $\nabla f = (f_x, f_y, f_z)$ of f is not $(0,0,0)$ at p

then, S is a smooth surface.

We will omit the proof here, but mention that it is a consequence of the inverse function theorem.

Ex. If S is the sphere of radius R centered at $(0,0,0)$, then we can take $W = \mathbb{R}^3$ and use:

$$f(x, y, z) = x^2 + y^2 + z^2 - R^2.$$

Then, $\nabla f = (2x, 2y, 2z) \neq (0,0,0)$ for any point of S (since $(0,0,0)$ is not a point of S). Hence, S is a smooth surface.

Ex. Let S be the circular cone $y^2 = x^2 + z^2$. Again, we can take $W = \mathbb{R}^3$ and let:

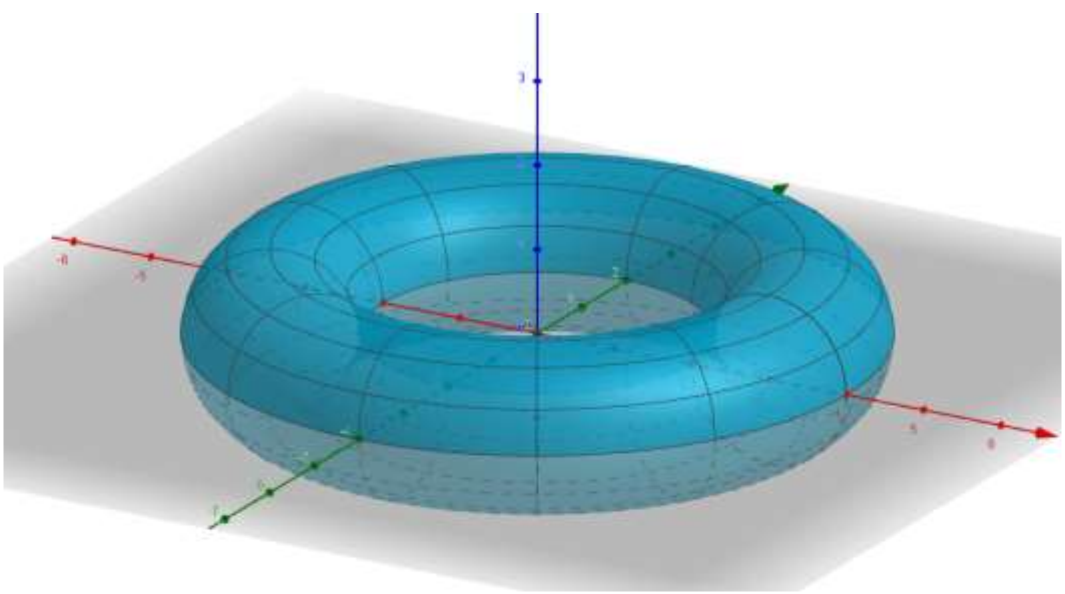
$$f(x, y, z) = x^2 + z^2 - y^2.$$

$\nabla f = (2x, -2y, 2z)$ but $\nabla f(0,0,0) = (0,0,0)$, and $(0,0,0)$ is a point of S. However, if we remove $(0,0,0)$ from S, then it is a smooth surface.

Not every surface can be represented by a single equation in x, y, and z. For example, the following is a parametrization of a torus:

$$\vec{\Phi}(u, v) = \big((3 + \cos v)\cos u\,, (3 + \cos v)\sin u\,, \sin v\big);$$

where: $(u, v) \in [0, 2\pi] \times [0, 2\pi]$.



The upper half of the torus $(z \geq 0)$ can be represented by:

$$x^2 + y^2 = \left(3 + \sqrt{1 - z^2}\right)^2 \quad \text{since}$$

$$x^2 + y^2 = (3 + \cos v)^2 = \left(3 + \sqrt{1 - z^2}\right)^2; \quad z \geq 0$$

because $cosv = \pm\sqrt{1 - \sin^2 v}.$

While the bottom half $(z \leq 0)$ can be represented by:

$$x^2 + y^2 = \left(3 - \sqrt{1 - z^2}\right)^2.$$

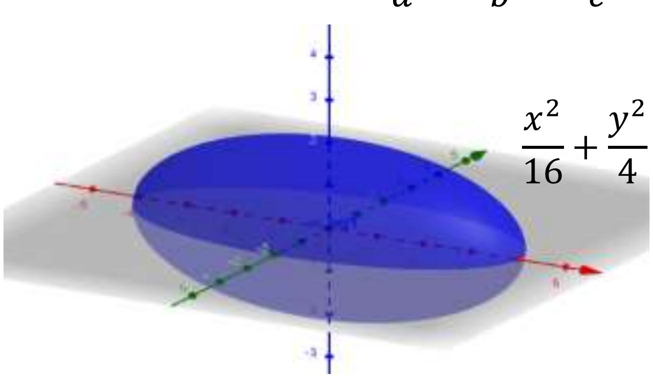
<u>Quadric Surfaces</u>

Quadric surfaces are generalizations of conic sections (circle, ellipse, hyperbola, and parabola) from $\mathbb{R}^2$ to $\mathbb{R}^3$. Thus, a quadric surface is the set of points that satisfies:

$$a_1x^2 + a_2y^2 + a_3z^2 + 2a_4xy + 2a_5yz + 2a_6xz + a_7x + a_8y + a_9z + a_{10} = 0.$$

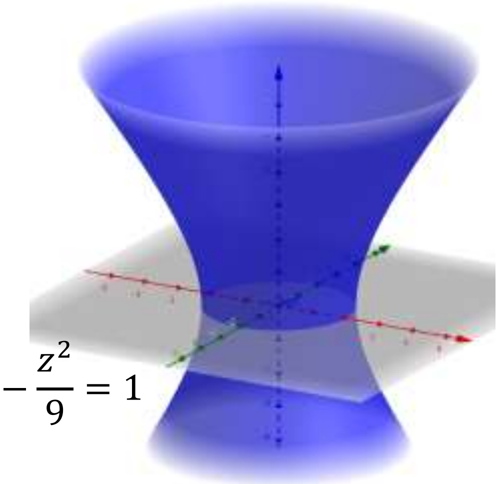
Not all quadric surfaces are surfaces. For example, the set of points that satisfy $x^2 + y^2 + z^2 = 0$ is just the point $(0,0,0)$. However, the quadric surfaces (other than planes) that actually are surfaces are the following:

i) Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



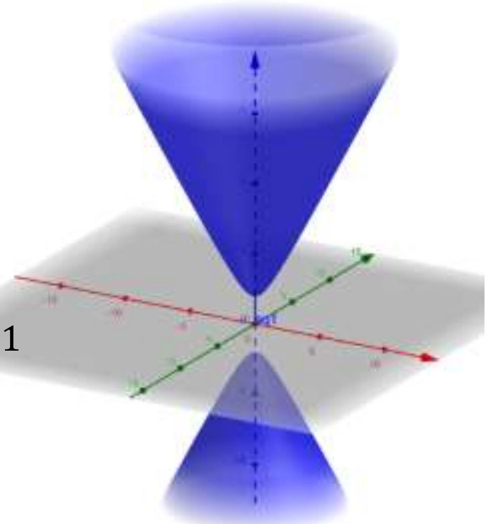
$$\frac{x^2}{16} + \frac{y^2}{4} + \frac{z^4}{4} = 1$$

ii) Hyperboloid of one sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



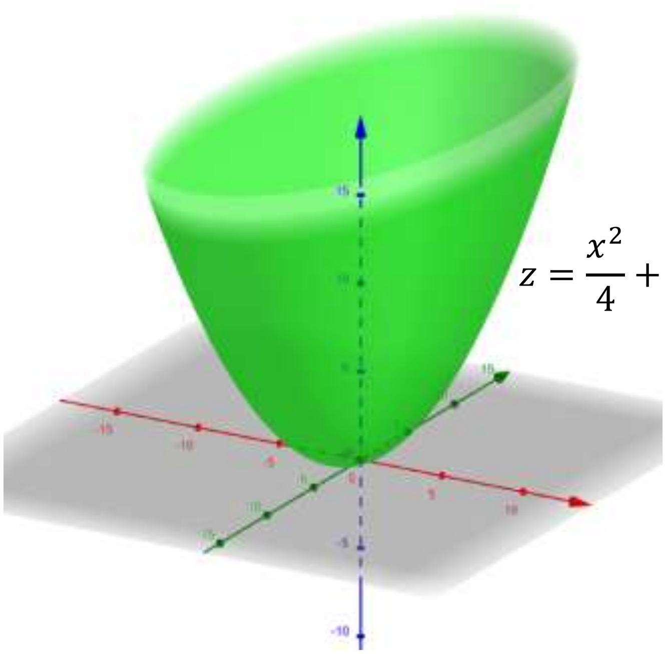
$$\frac{x^2}{4} + \frac{y^2}{4} - \frac{z^2}{9} = 1$$

iii) Hyperboloid of two sheets: $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



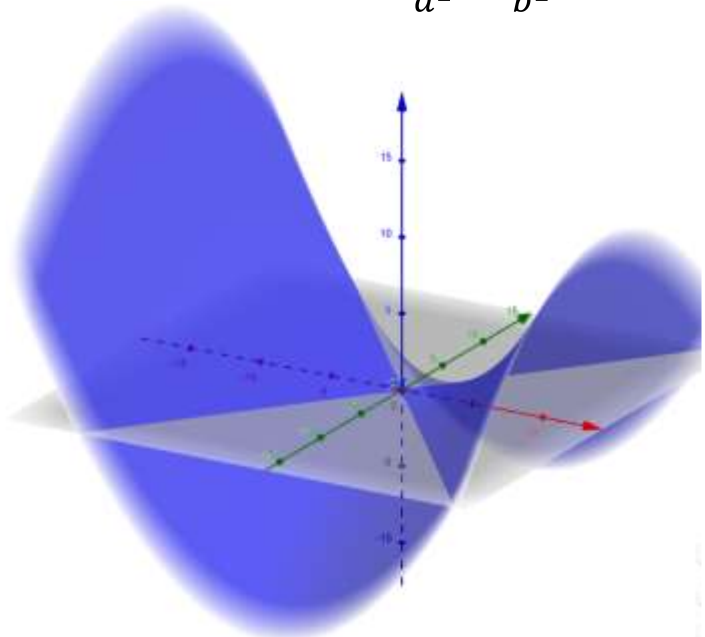
$$\frac{z^2}{4} - \frac{x^2}{1} - \frac{y^2}{1} = 1$$

iv) Elliptic paraboloid: $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$



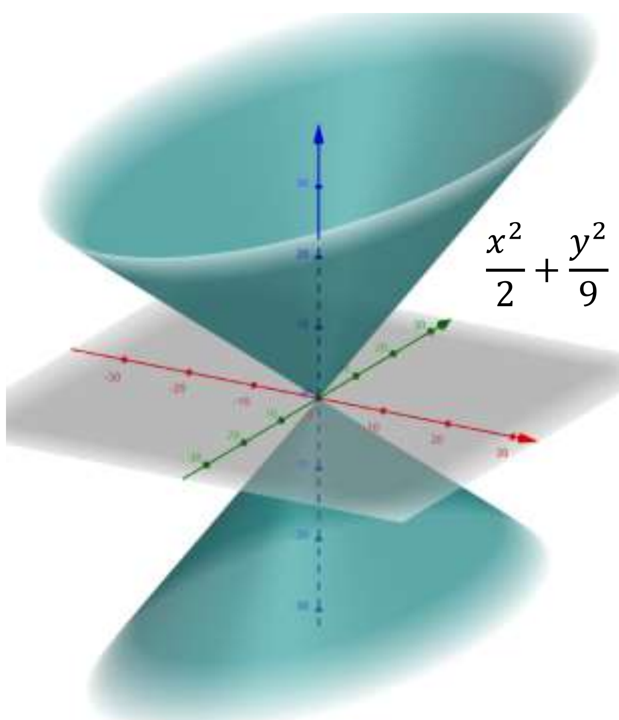
$$z = \frac{x^2}{4} + \frac{y^2}{16}$$

v) Hyperbolic paraboloid: $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ (saddle)



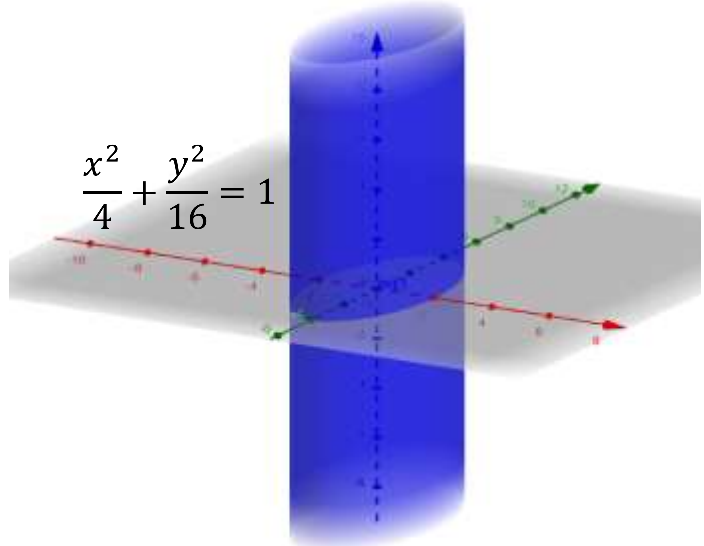
$$z = \frac{x^2}{16} - \frac{y^2}{25}$$

vi) Quadric Cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ (with $(0,0,0)$ removed)

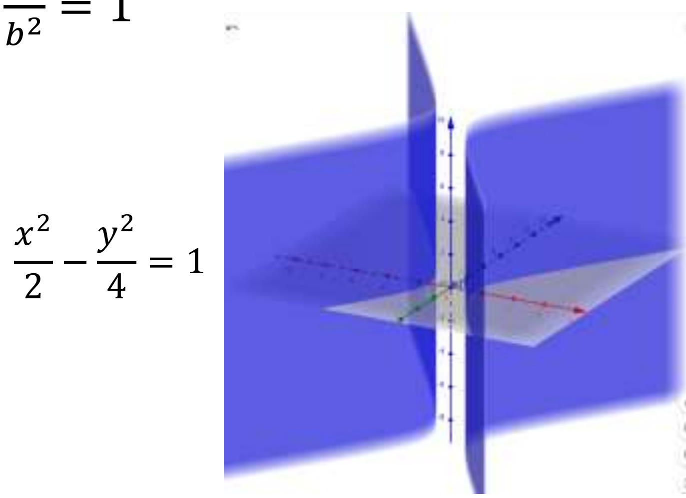


$$\frac{x^2}{2} + \frac{y^2}{9} - \frac{z^2}{4} = 0$$

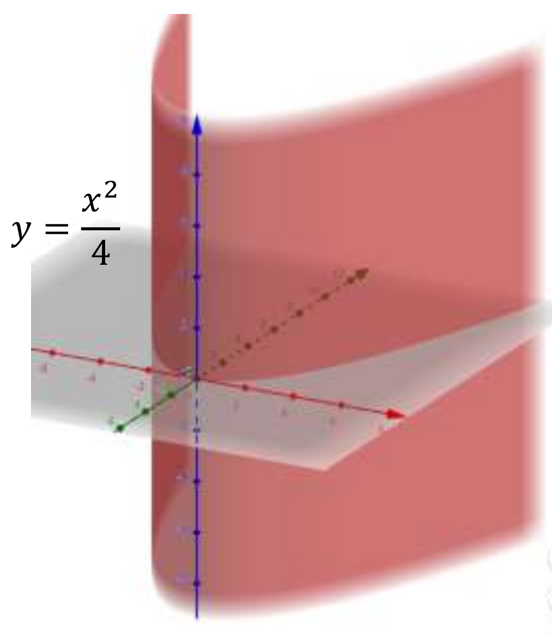
vii) Elliptic cylinder: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



viii) Hyperbolic cylinder: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



ix) Parabolic cylinder: $y = \frac{x^2}{a^2}$



All of these sets can be shown to be surfaces by transforming the equations into $f(x, y, z) = 0$ and showing $\nabla f \neq (0,0,0)$ for any points in the set. For example, for the two sheeted hyperboloid $\frac{z^2}{r^2} - \frac{x^2}{p^2} - \frac{y^2}{q^2} = 1$, let:

$$f(x, y, z) = \frac{z^2}{r^2} - \frac{x^2}{p^2} - \frac{y^2}{q^2} - 1.$$

Here we can take W in our earlier theorem to be $\mathbb{R}^3$.

$\nabla f = \left(\frac{-2x}{p^2}, \frac{-2y}{q^2}, \frac{2z}{r^2}\right) = (0,0,0)$ only when $x = 0, y = 0$, and $z = 0$, which is not a point on $\frac{z^2}{r^2} - \frac{x^2}{p^2} - \frac{y^2}{q^2} = 1$. Thus, this is a smooth surface.

Ex. Identify the following quadric surface and find a parametrization for it:

$$x^2 + 2y^2 + 6x - 4y - 3z = 1.$$

Complete the square:

$$(x^2 + 6x + \_\_\_\_) + 2(y^2 - 2y + \_\_\_\_) = 3z + 1$$

$$(x^2 + 6x + 9) + 2(y^2 - 2y + 1) = 3z + 1 + 9 + 2$$

$$(x + 3)^2 + 2(y - 1)^2 = 3z + 12$$

$$(x + 3)^2 + 2(y - 1)^2 = 3(z + 4).$$

If we let $x' = x + 3,\ y' = y - 1,\ z' = z + 4$, we get:

$$(x')^2 + 2(y')^2 = 3z'$$

or

$$\frac{(x')^2}{3} + \frac{(y')^2}{\frac{3}{2}} = z'.$$

This is an elliptic paraboloid.

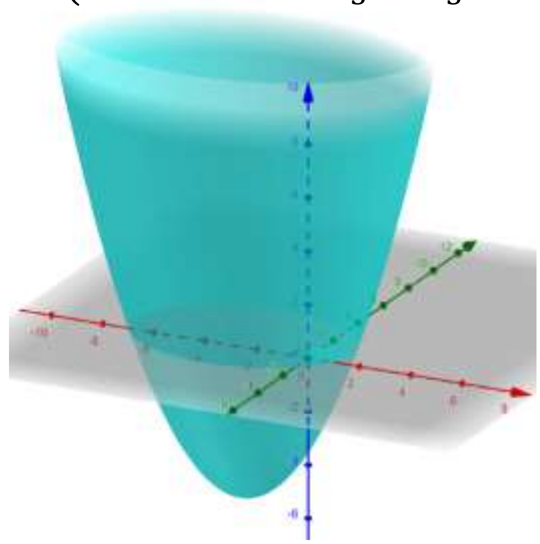
Since this is in the form $z' = f(x', y')$, we can parametrize this by:

$$x + 3 = x' = u$$

$$y - 1 = y' = v$$

$$z + 4 = z' = \frac{u^2}{3} + \frac{2v^2}{3} \quad \text{or}$$

$$\vec{\Phi}(u, v) = \left(u - 3,\ v + 1,\ \frac{u^2}{3} + \frac{2v^2}{3} - 4\right); \quad u, v \in \mathbb{R}.$$



Ex. Identify each of the following quadric surfaces

i) $\vec{\Phi}(u, v) = (u - v, u + v, 2u^2 + 2v^2)$

ii) $\vec{\Phi}(u, v) = (\cosh u, \sinh u, v)$

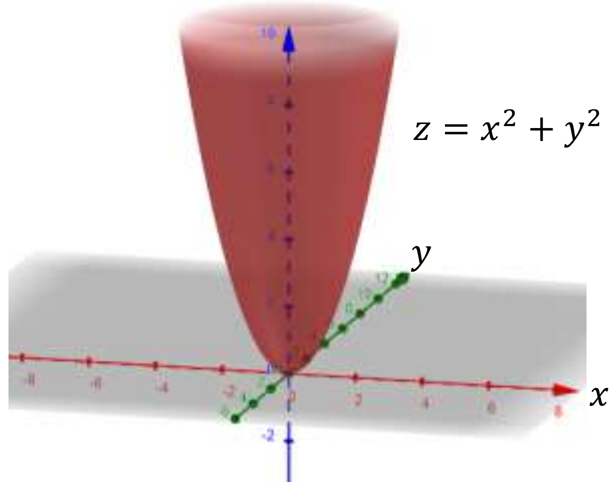
iii) $\vec{\Phi}(u, v) = \big((\sinh(u)\sinh(v), (\sinh(u))\cosh(v), \sinh(v)\big)$

i) $x = u - v \quad \Rightarrow \quad x + y = 2u \quad \Rightarrow \quad u = \frac{x+y}{2}$

$y = u + v \quad \Rightarrow \quad x - y = -2v \quad \Rightarrow \quad v = \frac{y-x}{2}$

$z = 2u^2 + 2v^2$

$z = 2\left(\frac{x+y}{2}\right)^2 + 2\left(\frac{y-x}{2}\right)^2 = x^2 + y^2 \quad \Rightarrow$ elliptic paraboloid.



ii) $x = \cosh u$

$y = \sinh u$

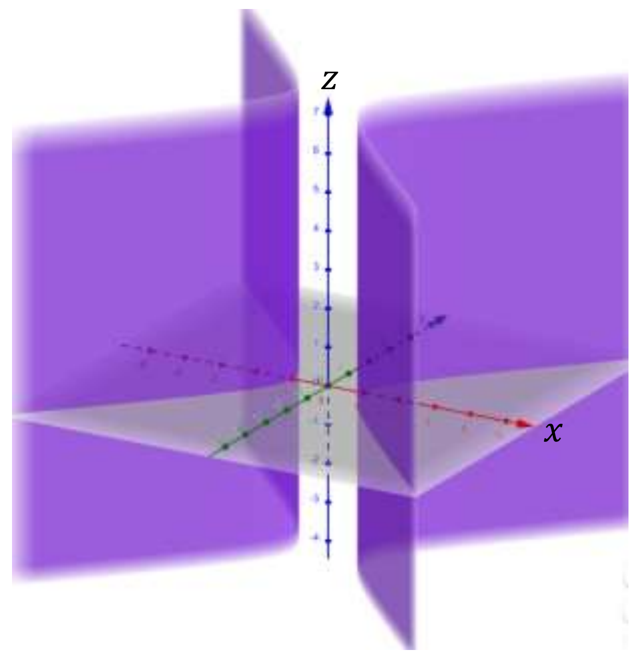
$z = v$

$\Rightarrow \quad x^2 - y^2 = \cosh^2 u - \sinh^2 u$

$= 1$

$x^2 - y^2 = 1$, z can take any value

$\Rightarrow$ hyperbolic cylinder.



iii) $x = (\sinh u) \sinh v$

$y = (\sinh u) \cosh v$

$z = \sinh u$

$$y^2 - x^2 = (\sinh^2 u) \cosh^2 v - (\sinh^2 u) \sinh^2 v$$

$$= (\sinh^2 u)[\cosh^2 u - \sinh^2 v]$$

$$= \sinh^2 u$$

$$= z^2$$

$y^2 - x^2 = z^2$ or $y^2 = x^2 + z^2$ $\Rightarrow$ cone.