Curves in $\mathbb{R}^2$ and $\mathbb{R}^3$/Arc Length

One way to define a curve in the plane, $\mathbb{R}^2$, is by saying a curve, C, is the set of points:

$$C = \{(x, y) \in \mathbb{R}^2 |\ f(x, y) = c\}$$

Ex. a) $y = x^2 - 1$ (or $y - x^2 = -1$, here $f(x, y) = y - x^2$)

b) $x^2 + y^2 = 4$ (here $f(x, y) = x^2 + y^2$)

A second way to define a curve is with parametric equations.

Def. A **parameterized curve** in $\mathbb{R}^n$ is a map $\gamma: (a, b) \to \mathbb{R}^n$, for some a, b with $-\infty \le a < b \le \infty$, where $(a, b) = \{t \in \mathbb{R} |\ a < t < b\}$.

Note: even though the curve γ is a vector function, we will write γ instead of $\vec{\gamma}$.

Ex. Any curve, C_1, in $\mathbb{R}^2$ of the form $y = f(x)$ or curve, C_2, of the form $x = g(y)$ can be given in parametric form by:

$$C_1:\ \ \gamma_1(t) = \big(t, f(t)\big)$$

$$C_2:\ \ \gamma_2(t) = (g(t), t)$$

For example, $y = x^2$ can be parameterized by:

$$\gamma_1(t) = (t, t^2)\ ;\ -\infty < t < \infty$$

and $x = e^y$ can be parameterized by:

$$\gamma_2(t) = (e^t, t)\ ;\ -\infty < t < \infty.$$

It's worth noting that there are an infinite number of ways to parametrize the same curve:

Ex. The parabola $y = x^2$ can be parametrized by:

a) $\gamma_1(t) = (t, t^2)\,;\quad -\infty < t < \infty$
b) $\gamma_2(t) = (t^3, t^6)\,;\ -\infty < t < \infty$
c) $\gamma_3(t) = (2t-1, (2t-1)^2)\,;\ -\infty < t < \infty.$

In fact, $\gamma(t) = \left(\alpha(t), \left(\alpha(t)\right)^2\right)$ is a parametrization of $y = x^2$ as long as the range of $\alpha(t)$ is all real numbers.

However, notice that if we take: $\gamma_4(t) = (t^2, t^4)\,;\ -\infty < t < \infty$, we only get the portion of $y = x^2$ where $x \geq 0$ since $t^2 \geq 0$.

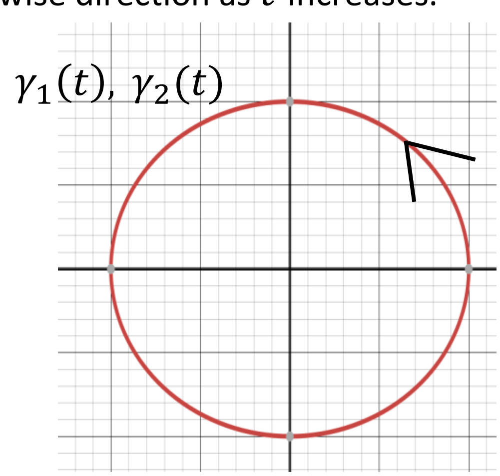
Ex. Parametrize the circle $x^2 + y^2 = 1$.

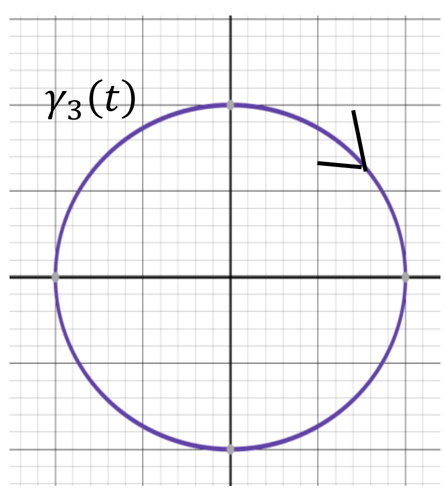
There are an infinite number of ways to do this. Here are some examples:

a) $\gamma_1(t) = (\cos t\,, \sin t)\,;\qquad -\infty < t < \infty$
b) $\gamma_2(t) = (\cos 2t\,, \sin 2t)\,;\ -\infty < t < \infty$
c) $\gamma_3(t) = (\sin t\,, \cos t)\,;\qquad -\infty < t < \infty.$

In each case the graph of $\gamma_i(t)$ goes around the circle an infinite number of times.

However, "a" and "b" move in a counterclockwise direction while "c" moves in a clockwise direction as t increases.



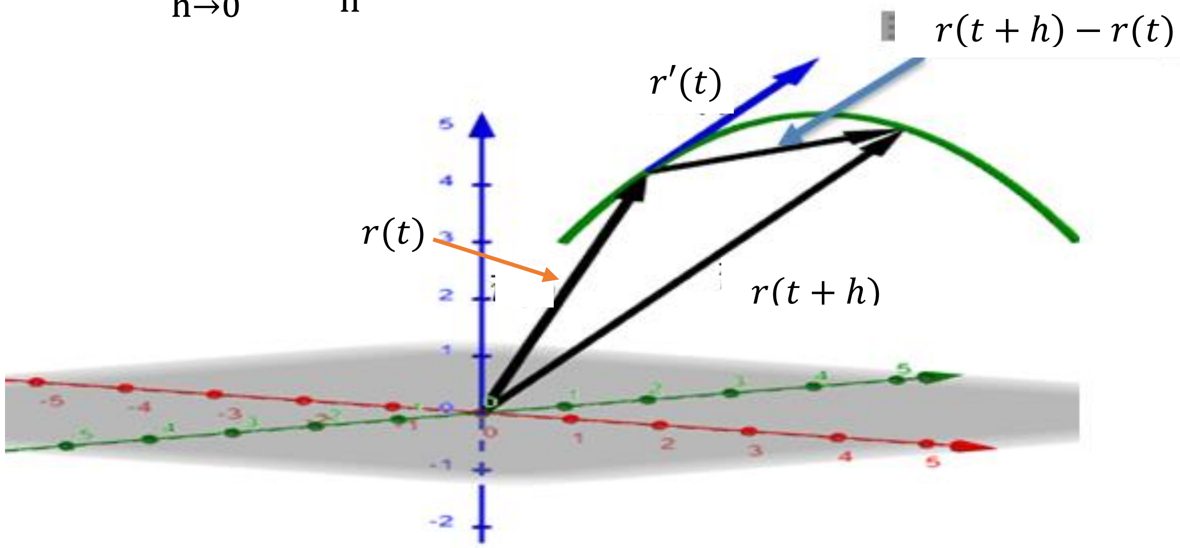


In general, we can define a curve in $\mathbb{R}^n$ by: $\gamma: (a, b) \to \mathbb{R}^n$

$$\gamma(t) = \big(\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t)\big).$$

We will restrict our attention to curves where $\frac{d^m \gamma_i(t)}{dt^m}$ exists for all $m \geq 1$ and all $t \in (a, b)$. These are called **smooth curves**.

Recall that the first derivative of a curve, $r(t)$, $r'(t) = \big(r'_1(t), \dots, r'_n(t)\big)$, is a tangent vector to $r(t)$. In vector form this is $r'(t) = \lim\limits_{h \to 0} \frac{r(t+h) - r(t)}{h}$



Ex. Find tangent vectors to the following curves at $t = \frac{\pi}{2}$.

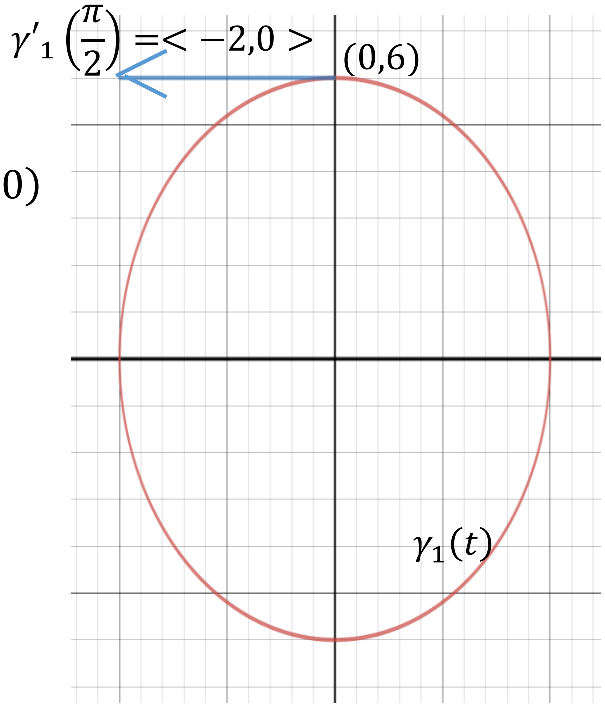
a) $\gamma_1(t) = (2\cos t, 6\sin t)$ (ellipse)

b) $\gamma_2(t) = (2\cos t, 2\sin t, 5t)$ (helix)

a) $\gamma'_1(t) = (-2\sin t, 6\cos t)$;

at $t = \frac{\pi}{2}$ we get:

$$\gamma'_1\left(\frac{\pi}{2}\right) = \left(-2\sin\frac{\pi}{2}, 6\cos\frac{\pi}{2}\right) = (-2, 0)$$



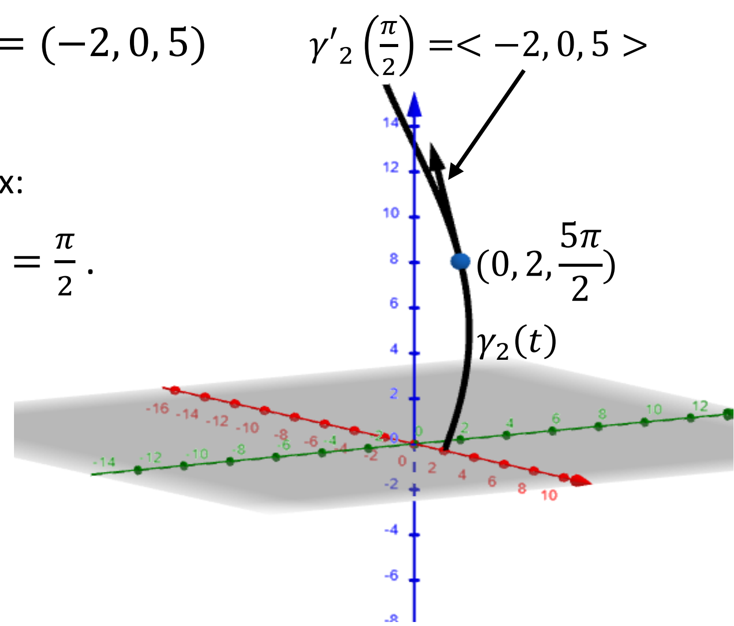
So $(-2, 0)$ is the tangent vector to the ellipse:

$$\gamma_1(t) = (2\cos t, 6\sin t) \text{ at } t = \frac{\pi}{2}\ .$$

$$\gamma_1\left(\frac{\pi}{2}\right) = \left(2\cos\frac{\pi}{2}, 6\sin\frac{\pi}{2}\right) = (0,6).$$

b) $\gamma'_2(t) = (-2\sin t, 2\cos t, 5)$

$$\gamma'_2\left(\frac{\pi}{2}\right) = \left(-2\sin\frac{\pi}{2}, 2\cos\frac{\pi}{2}, 5\right) = (-2, 0, 5)$$



So $(-2, 0, 5)$ is the tangent vector to the helix:

$$\gamma_2(t) = (2\cos t, 2\sin t, 5t) \text{ at } t = \frac{\pi}{2}.$$

$$\gamma_2\left(\frac{\pi}{2}\right) = (0,\ 2,\ \frac{5\pi}{2}).$$

Notice that there is more information in a parametrized curve than there is if the curve is given as $y = f(x)$ or $x = g(y)$.

Ex. $x^2 + y^2 = 1$, $\gamma_1(t) = (\cos t, \sin t)$, $\gamma_2(t) = (\cos 2t, \sin 2t)$ all have the same graph in $\mathbb{R}^2$. However, if we think of $\gamma_1(t)$ and $\gamma_2(t)$ as describing the path along which a particle is moving, then $\gamma_1(t)$ and $\gamma_2(t)$ not only tell us what points are on the graph but also the velocity (and acceleration) at any point on the graph.

Velocity vector$=\ \gamma'(t)$

Velocity vector$=\gamma'_1(t) = (-\sin t, \cos t)$

Velocity vector$=\gamma'_2(t) = (-2\sin 2t, 2\cos 2t)$

We define $\|\gamma'(t)\| =$ **speed** of $\gamma(t)$ at t.

$$\|\gamma'_1(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$$

$$\|\gamma'_2(t)\| = \sqrt{(-2\sin 2t)^2 + (2\cos 2t)^2} = 2$$

So γ_1 and γ_2 describe objects moving at different speeds along the same path.

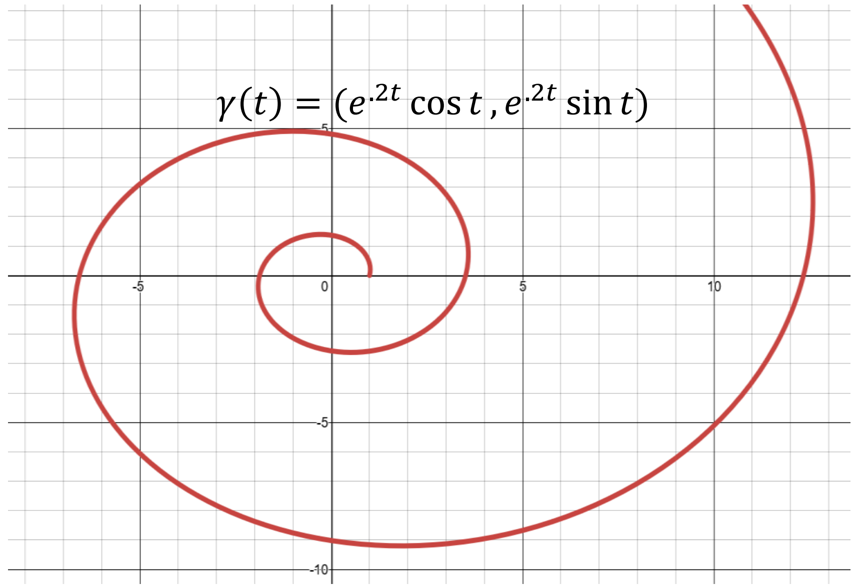
Def. The **arc length** of a curve γ starting at $\gamma(t_0)$ is given by:

$$s(t) = \int_{t_0}^{t} \|\gamma'(u)\|\ du.$$

Ex. Find the arc length function for the logarithmic spiral:

$\gamma(t) = (e^{kt}\cos t\,, e^{kt}\sin t)$

for $t > 0, k \neq 0$ a constant.



$$\gamma'(t) = (-e^{kt}\sin t + ke^{kt}\cos t\,, e^{kt}\cos t + ke^{kt}\sin t)$$

$$\gamma'(t) = (e^{kt}(k\cos t - \sin t),\ \ e^{kt}(k\sin t + \cos t))$$

$$\|\gamma'(t)\|^2 = e^{2kt}(k\cos t - \sin t)^2 + e^{2kt}(k\sin t + \cos t)^2$$

$$\|\gamma'(t)\|^2 = (k^2+1)e^{2kt}$$

$$\|\gamma'(t)\| = \left(\sqrt{k^2+1}\right)e^{kt}$$

$$s(t) = \int_{u=t_0}^{u=t} \sqrt{k^2+1}\, e^{ku}\, du = \frac{\sqrt{k^2+1}}{k}\, e^{ku}\Big|_{u=t_0}^{u=t}$$

$$= \frac{\sqrt{k^2+1}}{k}(e^{kt} - e^{kt_0})\,.$$

If we want to know the length of this curve between $t = 1$ and $t = 2$ we would get:

$$L = \int_{t=1}^{t=2} \sqrt{k^2+1}\, e^{kt}\, dt = \frac{\sqrt{k^2+1}}{k}\, e^{kt}\Big|_{t=1}^{t=2}$$

$$= \frac{\sqrt{k^2+1}}{k}(e^{2k} - e^{k})\,.$$

Notice that the arc length function, $s(t) = \int_{t_0}^{t} \|\gamma'(u)\| \; du$, has the property (by the Fundamental Theorem of Calculus) that:

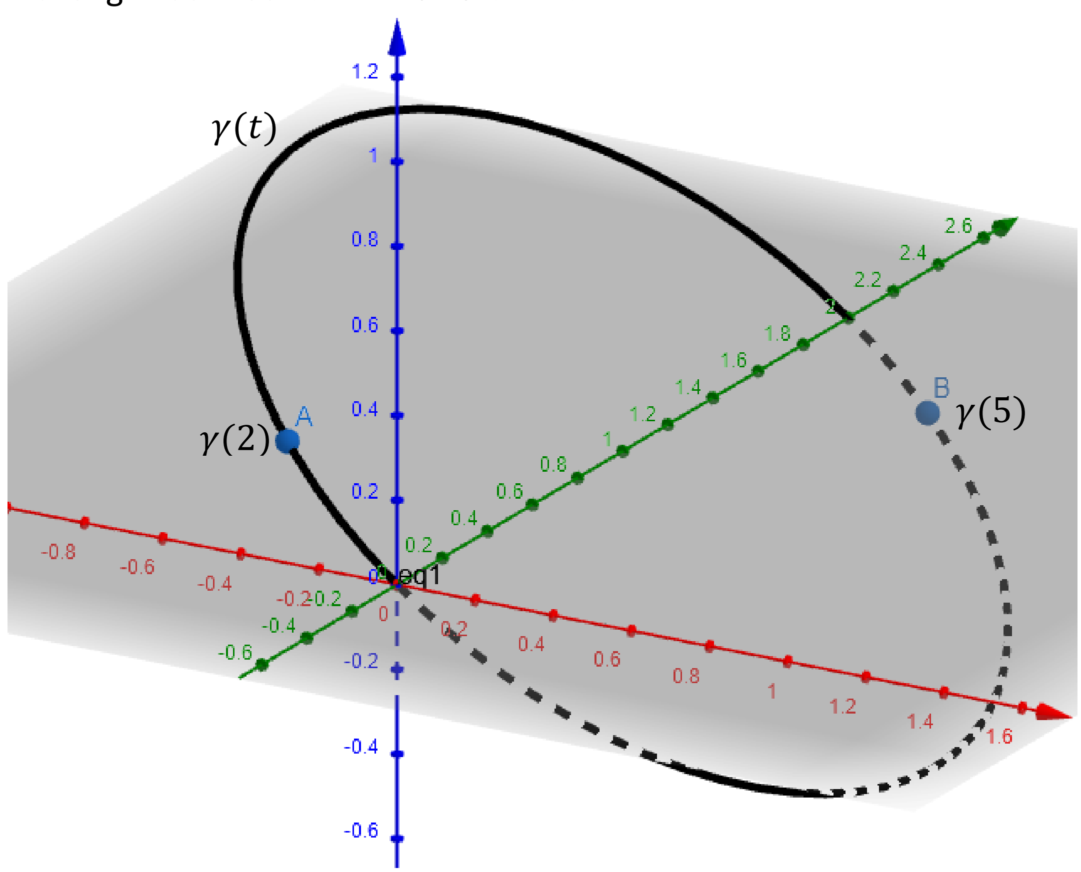
$$\frac{ds}{dt} = \|\gamma'(t)\|.$$

Def. If $\gamma\colon (a, b) \to \mathbb{R}^n$ is a parametrized curve, its speed at the point $\gamma(t)$ is $\|\gamma'(t)\|$, and γ is said to be a **unit speed curve** if $\gamma'(t)$ is a unit vector for all $t \in (a, b)$.

Ex. $\gamma(t) = (\cos t, \sin t)$ is a unit speed curve since

$\gamma'(t) = (-\sin t, \cos t)$ and $\|\gamma'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1.$

Ex. Show $\gamma(t) = \left(\frac{4}{5}\cos t\,, 1 - \sin t\,, -\frac{3}{5}\cos t\right)$ has unit speed and find its length between $t = 2$ and $t = 5$.



$$\gamma'(t) = (-\tfrac{4}{5}\sin t\,, -\cos t\,, \tfrac{3}{5}\sin t)$$

$$\|\gamma'(t)\| = \sqrt{\left(-\tfrac{4}{5}\sin t\right)^2 + (-\cos t)^2 + \left(\tfrac{3}{5}\sin t\right)^2}$$

$$= \sqrt{\frac{16}{25}\sin^2 t + \cos^2 t + \frac{9}{25}\sin^2 t}$$

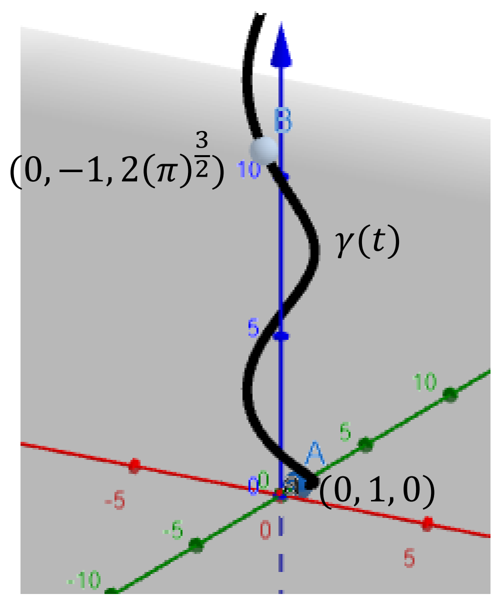
$$= \sqrt{\sin^2 t + \cos^2 t} = 1$$

So $\|\gamma'(t)\| = 1$ and $\gamma(t)$ has unit speed.

$L = \int_{t=2}^{t=5} \|\gamma'(t)\| dt = \int_{t=2}^{t=5} dt = 5 - 2 = 3\,.$

The length of any unit speed curve, $\gamma(t)$, between $t = a$ and $t = b$, where $b \geq a$, will always be $b - a$.

Ex. Find the length of $\gamma(t) = \left(\sin 3t\,,\ cos3t,\ 2t^{\frac{3}{2}}\right)$ between the points $(0,1,0)$ and $(0,-1,2(\pi)^{\frac{3}{2}})$.



First let's determine which values of t correspond to $(0,1,0)$ and $(0,-1,2(\pi)^{\frac{3}{2}})$.

$$\left(\sin 3t\,,\ cos3t,\ 2t^{\frac{3}{2}}\right) = (0,1,0)$$
$$\Rightarrow \quad t = 0.$$

$$\left(\sin 3t\,,\ cos3t,\ 2t^{\frac{3}{2}}\right) = (0,-1,2(\pi)^{\frac{3}{2}})$$
$$\Rightarrow \quad t = \pi.$$

$$\gamma'(t) = (3\cos t, -3\sin t, 3t^{\frac{1}{2}})$$

$$\|\gamma'(t)\| = \sqrt{9\cos^2 t + 9\sin^2 t + 9t} = 3(1+t)^{\frac{1}{2}}.$$

$$L = \int_{t=0}^{t=\pi} \|\gamma'(t)\| dt = \int_{t=0}^{t=\pi} 3(1+t)^{\frac{1}{2}} dt$$

$$= 3\left(\frac{2}{3}\right)(1+t)^{\frac{3}{2}}\Big|_{t=0}^{t=\pi} = 2\left[(1+\pi)^{\frac{3}{2}} - 1\right].$$

As we have seen, there are many ways to parametrize a curve. However, when we parametrize a curve using arc length as the parameter we will find that this simplifies many of our calculations. This is because if $s =$ arc length is the parameter for a curve then the curve is unit speed, i.e., $\|\gamma'(s)\| = 1$.

Let's see why this is true. Notice that by the chain rule we have:

$\gamma(s) = \big(x(s), y(s), z(s)\big)$ and

$$\frac{d\gamma}{dt} = \left(\frac{dx}{ds}\frac{ds}{dt}, \frac{dy}{ds}\frac{ds}{dt}, \frac{dz}{ds}\frac{ds}{dt}\right) = \frac{d\gamma}{ds}\,\frac{ds}{dt}.$$

Remember that $\frac{ds}{dt}$ is a scalar, so if it's not 0 we can say:

$\frac{d\gamma}{ds} = \frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}}$. Taking the length of both sides we get:

$$\left\|\frac{d\gamma}{ds}\right\| = \left\|\frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}}\right\| = \frac{\left\|\frac{d\gamma}{dt}\right\|}{\left|\frac{ds}{dt}\right|};$$

but we saw earlier that $\frac{ds}{dt} = \left\|\gamma'(t)\right\| = \left\|\frac{d\gamma}{dt}\right\|$

Thus $\left\|\frac{d\gamma}{ds}\right\| = \|\gamma'(s)\| = 1.$

Recall that the dot product of two vectors in $\mathbb{R}^n$:

$\vec{a} = (a_1, a_2, \dots, a_n)$ and $\vec{b} = (b_1, b_2, \dots, b_n)$ is

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i.$$

If $\vec{a}(t), \vec{b}(t)$ are vector functions of t then:

$$\vec{a}(t) \cdot \vec{b}(t) = \sum_{i=1}^{n} a_i(t)\, b_i(t)$$

and

$$\frac{d}{dt}\left(\vec{a}(t) \cdot \vec{b}(t)\right) = \frac{d\vec{a}(t)}{dt} \cdot \vec{b}(t) + \vec{a}(t) \cdot \frac{d\vec{b}(t)}{dt}$$

Since: $\frac{d}{dt}\left(a_i(t) b_i(t)\right) = \frac{da_i(t)}{dt} \cdot b_i(t) + a_i(t) \cdot \frac{db_i(t)}{dt}$.

Proposition: Suppose $\vec{n}(t)$ is a smooth vector function of t and $\|\vec{n}(t)\| = 1$ for all t, then $\vec{n}'(t)$ and $\vec{n}(t)$ are perpendicular to each other, thus $\vec{n}'(t) \cdot \vec{n}(t) = 0$ for all t.

Proof:

$$\vec{n}(t) \cdot \vec{n}(t) = \|\vec{n}\|^2 = 1$$

$$\frac{d\vec{n}}{dt} \cdot \vec{n} + \vec{n} \cdot \frac{d\vec{n}}{dt} = 0 \quad \text{for all } t$$

or

$$\vec{n}'(t) \cdot \vec{n}(t) = 0 \quad \text{for all } t.$$