

Curves in \mathbb{R}^2 and \mathbb{R}^3 /Arc Length

One way to define a curve in the plane, \mathbb{R}^2 , is by saying a curve, C , is the set of points:

$$C = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}$$

Ex. a) $y = x^2 - 1$ (or $y - x^2 = -1$, here $f(x, y) = y - x^2$)

b) $x^2 + y^2 = 4$ (here $f(x, y) = x^2 + y^2$)

A second way to define a curve is with parametric equations.

Def. A **parameterized curve** in \mathbb{R}^n is a map $\gamma: (a, b) \rightarrow \mathbb{R}^n$, for some a, b with $-\infty \leq a < b \leq \infty$, where $(a, b) = \{t \in \mathbb{R} \mid a < t < b\}$.

Note: even though the curve γ is a vector function, we will write γ instead of $\vec{\gamma}$.

Ex. Any curve, C_1 , in \mathbb{R}^2 of the form $y = f(x)$ or curve, C_2 , of the form

$x = g(y)$ can be given in parametric form by:

$$C_1: \gamma_1(t) = (t, f(t))$$

$$C_2: \gamma_2(t) = (g(t), t)$$

For example, $y = x^2$ can be parameterized by:

$$\gamma_1(t) = (t, t^2); -\infty < t < \infty$$

and $x = e^y$ can be parameterized by:

$$\gamma_2(t) = (e^t, t); -\infty < t < \infty.$$

It's worth noting that there are an infinite number of ways to parametrize the same curve:

Ex. The parabola $y = x^2$ can be parametrized by:

a) $\gamma_1(t) = (t, t^2); \quad -\infty < t < \infty$

b) $\gamma_2(t) = (t^3, t^6); \quad -\infty < t < \infty$

c) $\gamma_3(t) = (2t - 1, (2t - 1)^2); \quad -\infty < t < \infty.$

In fact, $\gamma(t) = (\alpha(t), (\alpha(t))^2)$ is a parametrization of $y = x^2$ as long as the range of $\alpha(t)$ is all real numbers.

However, notice that if we take: $\gamma_4(t) = (t^2, t^4); \quad -\infty < t < \infty$, we only get the portion of $y = x^2$ where $x \geq 0$ since $t^2 \geq 0$.

Ex. Parametrize the circle $x^2 + y^2 = 1$.

There are an infinite number of ways to do this. Here are some examples:

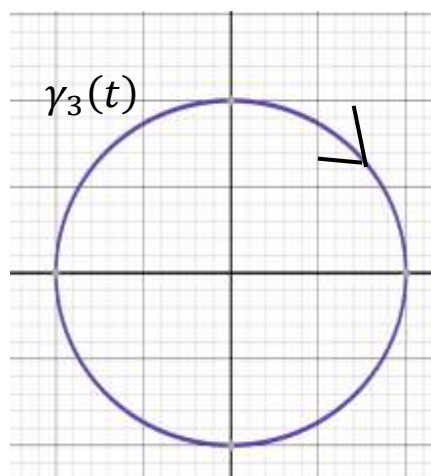
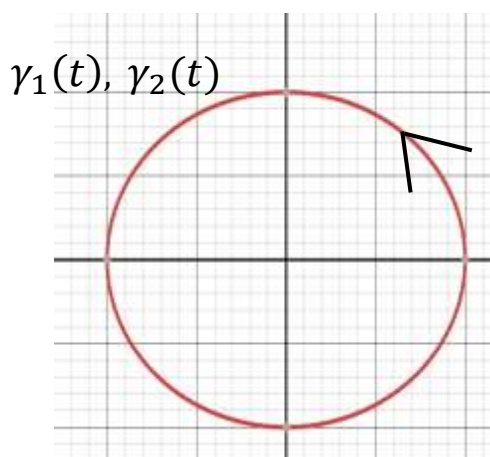
a) $\gamma_1(t) = (\cos t, \sin t); \quad -\infty < t < \infty$

b) $\gamma_2(t) = (\cos 2t, \sin 2t); \quad -\infty < t < \infty$

c) $\gamma_3(t) = (\sin t, \cos t); \quad -\infty < t < \infty.$

In each case the graph of $\gamma_i(t)$ goes around the circle an infinite number of times.

However, "a" and "b" move in a counterclockwise direction while "c" moves in a clockwise direction as t increases.



In general, we can define a curve in \mathbb{R}^n by: $\gamma: (a, b) \rightarrow \mathbb{R}^n$

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t)).$$

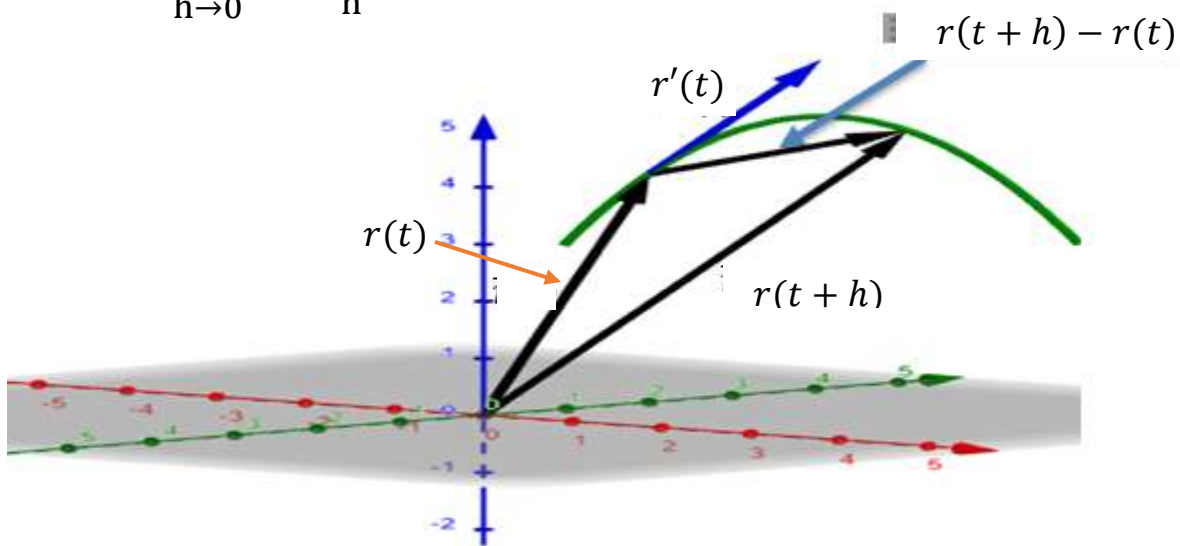
We will restrict our attention to curves where $\frac{d^m \gamma_i(t)}{dt^m}$ exists for all $m \geq 1$

and all $t \in (a, b)$. These are called **smooth curves**.

Recall that the first derivative of a curve, $r'(t)$,

$r'(t) = (r'_1(t), \dots, r'_n(t))$, is a tangent vector to $r(t)$. In vector form this

is $r'(t) = \lim_{h \rightarrow 0} \frac{r(t+h) - r(t)}{h}$



Ex. Find tangent vectors to the following curves at $t = \frac{\pi}{2}$.

a) $\gamma_1(t) = (2 \cos t, 6 \sin t)$ (ellipse)

b) $\gamma_2(t) = (2 \cos t, 2 \sin t, 5t)$ (helix)

a) $\gamma'_1(t) = (-2 \sin t, 6 \cos t)$;

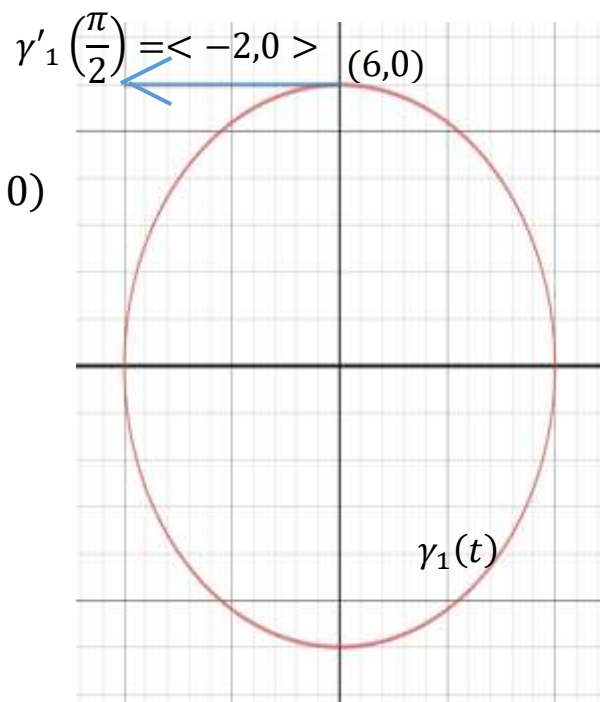
at $t = \frac{\pi}{2}$ we get:

$$\gamma'_1\left(\frac{\pi}{2}\right) = \left(-2 \sin \frac{\pi}{2}, 6 \cos \frac{\pi}{2}\right) = (-2, 0)$$

So $(-2, 0)$ is the tangent vector to the ellipse:

$$\gamma_1(t) = (2 \cos t, 6 \sin t) \text{ at } t = \frac{\pi}{2}.$$

$$\gamma_1\left(\frac{\pi}{2}\right) = \left(2 \cos \frac{\pi}{2}, 6 \sin \frac{\pi}{2}\right) = (0, 6).$$



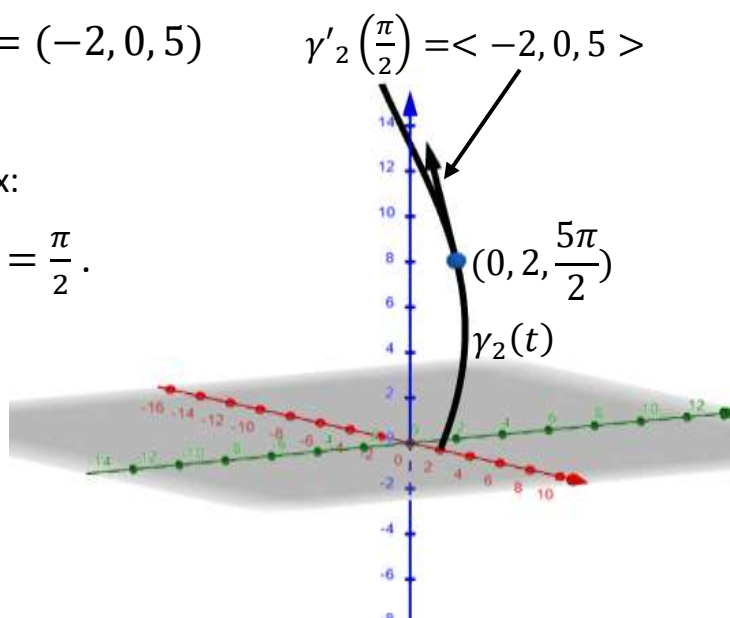
b) $\gamma'_2(t) = (-2 \sin t, 2 \cos t, 5)$

$$\gamma'_2\left(\frac{\pi}{2}\right) = \left(-2 \sin \frac{\pi}{2}, 2 \cos \frac{\pi}{2}, 5\right) = (-2, 0, 5)$$

So $(-2, 0, 5)$ is the tangent vector to the helix:

$$\gamma_2(t) = (2 \cos t, 2 \sin t, 5t) \text{ at } t = \frac{\pi}{2}.$$

$$\gamma_2\left(\frac{\pi}{2}\right) = \left(0, 2, \frac{5\pi}{2}\right).$$



Notice that there is more information in a parametrized curve than there is if the curve is given as $y = f(x)$ or $x = g(y)$.

Ex. $x^2 + y^2 = 1$, $\gamma_1(t) = (\cos t, \sin t)$, $\gamma_2(t) = (\cos 2t, \sin 2t)$ all have the same graph in \mathbb{R}^2 . However, if we think of $\gamma_1(t)$ and $\gamma_2(t)$ as describing the path along which a particle is moving, then $\gamma_1(t)$ and $\gamma_2(t)$ not only tell us what points are on the graph but also the velocity (and acceleration) at any point on the graph.

$$\mathbf{Velocity\ vector} = \gamma'(t)$$

$$\mathbf{Velocity\ vector} = \gamma'_1(t) = (-\sin t, \cos t)$$

$$\mathbf{Velocity\ vector} = \gamma'_2(t) = (-2 \sin 2t, 2 \cos 2t)$$

We define $\|\gamma'(t)\| = \mathbf{speed}$ of $\gamma(t)$ at t .

$$\|\gamma'_1(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$$

$$\|\gamma'_2(t)\| = \sqrt{(-2 \sin 2t)^2 + (2 \cos 2t)^2} = 2$$

So γ_1 and γ_2 describe objects moving at different speeds along the same path.

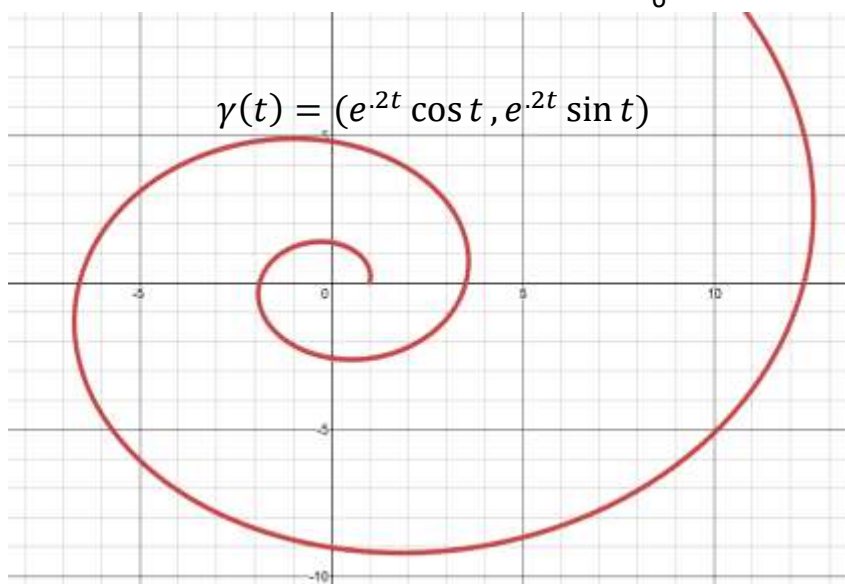
Def. The **arc length** of a curve γ starting at $\gamma(t_0)$ is given by:

$$s(t) = \int_{t_0}^t \|\gamma'(u)\| \, du.$$

Ex. Find the arc length function
for the logarithmic spiral:

$$\gamma(t) = (e^{kt} \cos t, e^{kt} \sin t)$$

for $t > 0, k \neq 0$ a constant.



$$\gamma'(t) = (-e^{kt} \sin t + ke^{kt} \cos t, e^{kt} \cos t + ke^{kt} \sin t)$$

$$\gamma'(t) = (e^{kt}(k \cos t - \sin t), e^{kt}(k \sin t + \cos t))$$

$$\|\gamma'(t)\|^2 = e^{2kt}(k \cos t - \sin t)^2 + e^{2kt}(k \sin t + \cos t)^2$$

$$\|\gamma'(t)\|^2 = (k^2 + 1)e^{2kt}$$

$$\|\gamma'(t)\| = (\sqrt{k^2 + 1})e^{kt}$$

$$\begin{aligned} s(t) &= \int_{u=t_0}^{u=t} \sqrt{k^2 + 1} e^{ku} du = \frac{\sqrt{k^2 + 1}}{k} e^{kt} \Big|_{u=t_0}^{u=t} \\ &= \frac{\sqrt{k^2 + 1}}{k} (e^{kt} - e^{kt_0}). \end{aligned}$$

If we want to know the length of this curve between $t = 1$ and $t = 2$
we would get:

$$\begin{aligned} L &= \int_{t=1}^{t=2} \sqrt{k^2 + 1} e^{kt} dt = \frac{\sqrt{k^2 + 1}}{k} e^{kt} \Big|_{t=1}^{t=2} \\ &= \frac{\sqrt{k^2 + 1}}{k} (e^{2k} - e^k). \end{aligned}$$

Notice that the arc length function, $s(t) = \int_{t_0}^t \|\gamma'(u)\| \, du$, has the property (by the Fundamental Theorem of Calculus) that:

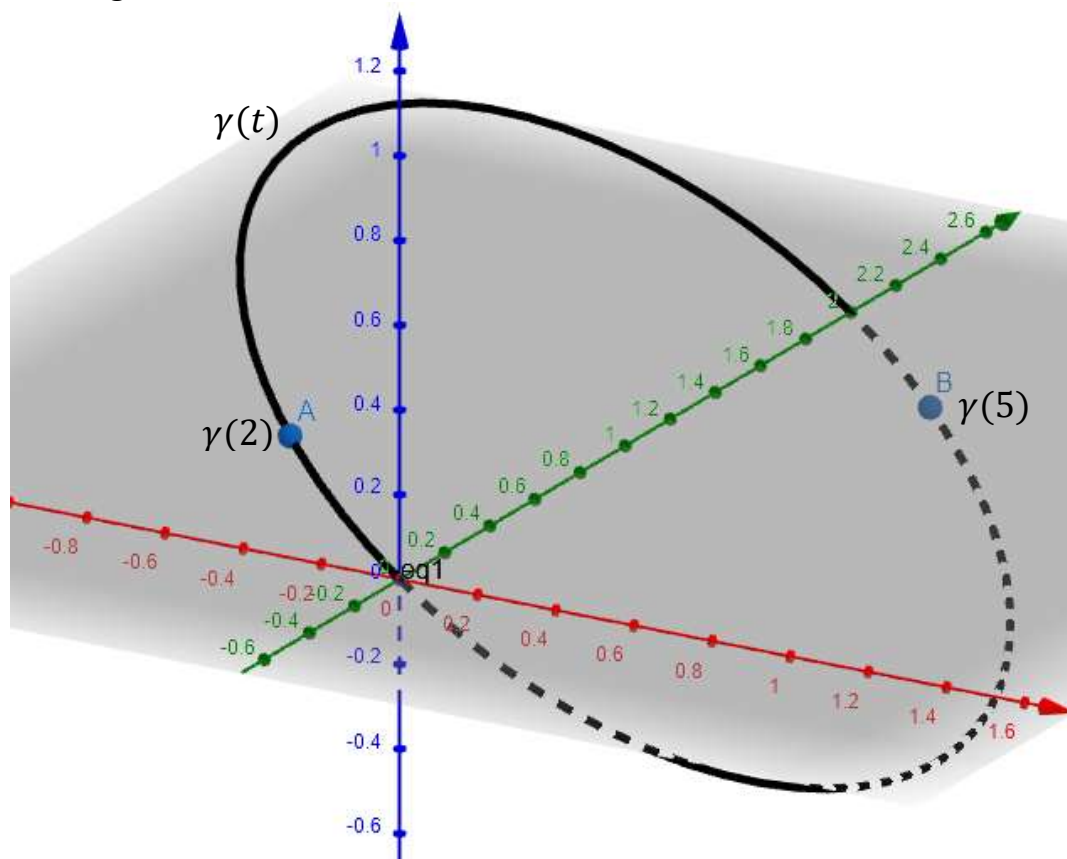
$$\frac{ds}{dt} = \|\gamma'(t)\|.$$

Def. If $\gamma: (a, b) \rightarrow \mathbb{R}^n$ is a parametrized curve, its speed at the point $\gamma(t)$ is $\|\gamma'(t)\|$, and γ is said to be a **unit speed curve** if $\gamma'(t)$ is a unit vector for all $t \in (a, b)$.

Ex. $\gamma(t) = (\cos t, \sin t)$ is a unit speed curve since

$$\gamma'(t) = (-\sin t, \cos t) \text{ and } \|\gamma'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1.$$

Ex. Show $\gamma(t) = \left(\frac{4}{5} \cos t, 1 - \sin t, -\frac{3}{5} \cos t\right)$ has unit speed and find its length between $t = 2$ and $t = 5$.



$$\gamma'(t) = \left(-\frac{4}{5}\sin t, -\cos t, \frac{3}{5}\sin t\right)$$

$$\begin{aligned}\|\gamma'(t)\| &= \sqrt{\left(-\frac{4}{5}\sin t\right)^2 + (-\cos t)^2 + \left(\frac{3}{5}\sin t\right)^2} \\ &= \sqrt{\frac{16}{25}\sin^2 t + \cos^2 t + \frac{9}{25}\sin^2 t} \\ &= \sqrt{\sin^2 t + \cos^2 t} = 1\end{aligned}$$

So $\|\gamma'(t)\| = 1$ and $\gamma(t)$ has unit speed.

$$L = \int_{t=2}^{t=5} \|\gamma'(t)\| dt = \int_{t=2}^{t=5} dt = 5 - 2 = 3.$$

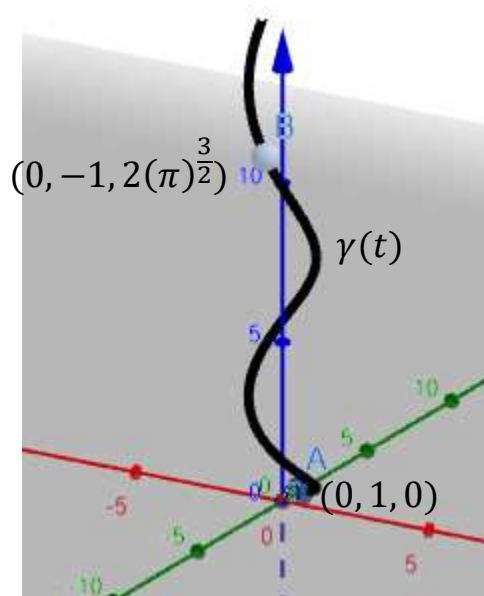
The length of any unit speed curve, $\gamma(t)$, between $t = a$ and $t = b$, where $b \geq a$, will always be $b - a$.

Ex. Find the length of $\gamma(t) = \left(\sin 3t, \cos 3t, 2t^{\frac{3}{2}}\right)$ between the points $(0, 1, 0)$ and $(0, -1, 2(\pi)^{\frac{3}{2}})$.

First let's determine which values of t correspond to $(0, 1, 0)$ and $(0, -1, 2(\pi)^{\frac{3}{2}})$.

$$\begin{aligned}\left(\sin 3t, \cos 3t, 2t^{\frac{3}{2}}\right) &= (0, 1, 0) \\ \Rightarrow t &= 0.\end{aligned}$$

$$\begin{aligned}\left(\sin 3t, \cos 3t, 2t^{\frac{3}{2}}\right) &= (0, -1, 2(\pi)^{\frac{3}{2}}) \\ \Rightarrow t &= \pi.\end{aligned}$$



$$\gamma'(t) = (3 \cos t, -3 \sin t, 3t^{\frac{1}{2}})$$

$$\|\gamma'(t)\| = \sqrt{9 \cos^2 t + 9 \sin^2 t + 9t} = 3(1+t)^{\frac{1}{2}}.$$

$$\begin{aligned} L &= \int_{t=0}^{t=\pi} \|\gamma'(t)\| dt = \int_{t=0}^{t=\pi} 3(1+t)^{\frac{1}{2}} dt \\ &= 3 \left(\frac{2}{3} \right) (1+t)^{\frac{3}{2}} \Big|_{t=0}^{t=\pi} = 2 \left[(1+\pi)^{\frac{3}{2}} - 1 \right]. \end{aligned}$$

As we have seen, there are many ways to parametrize a curve. However, when we parametrize a curve using arc length as the parameter we will find that this simplifies many of our calculations. This is because if $s = \text{arc length}$ is the parameter for a curve then the curve is unit speed, i.e., $\|\gamma'(s)\| = 1$.

Let's see why this is true. Notice that by the chain rule we have:

$$\gamma(s) = (x(s), y(s), z(s)) \text{ and}$$

$$\frac{d\gamma}{dt} = \left(\frac{dx}{ds} \frac{ds}{dt}, \frac{dy}{ds} \frac{ds}{dt}, \frac{dz}{ds} \frac{ds}{dt} \right) = \frac{d\gamma}{ds} \frac{ds}{dt}.$$

Remember that $\frac{ds}{dt}$ is a scalar, so if it's not 0 we can say:

$$\frac{d\gamma}{ds} = \frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}}. \text{ Taking the length of both sides we get:}$$

$$\left\| \frac{d\gamma}{ds} \right\| = \left\| \frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}} \right\| = \frac{\left\| \frac{d\gamma}{dt} \right\|}{\left| \frac{ds}{dt} \right|};$$

$$\text{but we saw earlier that } \frac{ds}{dt} = \|\gamma'(t)\| = \left\| \frac{d\gamma}{dt} \right\|$$

$$\text{Thus } \left\| \frac{d\gamma}{ds} \right\| = \|\gamma'(s)\| = 1.$$

Recall that the dot product of two vectors in \mathbb{R}^n :

$$\vec{a} = (a_1, a_2, \dots, a_n) \text{ and } \vec{b} = (b_1, b_2, \dots, b_n) \text{ is}$$
$$\vec{a} \cdot \vec{b} = \sum_{i=1}^n a_i b_i.$$

If $\vec{a}(t), \vec{b}(t)$ are vector functions of t then:

$$\vec{a}(t) \cdot \vec{b}(t) = \sum_{i=1}^n a_i(t) b_i(t)$$

and

$$\frac{d}{dt} (\vec{a}(t) \cdot \vec{b}(t)) = \frac{d\vec{a}(t)}{dt} \cdot \vec{b}(t) + \vec{a}(t) \cdot \frac{d\vec{b}(t)}{dt}$$

Since: $\frac{d}{dt} (a_i(t) b_i(t)) = \frac{da_i(t)}{dt} \cdot b_i(t) + a_i(t) \cdot \frac{db_i(t)}{dt}.$

Proposition: Suppose $\vec{n}(t)$ is a smooth vector function of t and $\|\vec{n}(t)\| = 1$ for all t , then $\vec{n}'(t)$ and $\vec{n}(t)$ are perpendicular to each other, thus $\vec{n}'(t) \cdot \vec{n}(t) = 0$ for all t .

Proof:

$$\vec{n}(t) \cdot \vec{n}(t) = \|\vec{n}\|^2 = 1$$

$$\frac{d\vec{n}}{dt} \cdot \vec{n} + \vec{n} \cdot \frac{d\vec{n}}{dt} = 0 \quad \text{for all } t$$

or

$$\vec{n}'(t) \cdot \vec{n}(t) = 0 \quad \text{for all } t.$$