Homogeneous Equations with Constant Coefficients

To solve an n^{th} order homogeneous linear differential equation with constant coefficients:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

$$a_0, a_1, \ldots, a_n \in \mathbb{R}$$

We guess at a solution: $y = e^{rx}$.

Given that the derivatives of y as

$$y' = re^{rx}$$

$$y'' = r^2 e^{rx}$$

$$\vdots$$

$$y^{(n)} = r^n e^{rx}$$

we get:

$$a_n r^n e^{rx} + \cdots + a_1 r e^{rx} + a_0 e^{rx} = 0$$

$$(a_n r^n + \cdots + a_1 r + a_0) e^{rx} = 0.$$

So we must solve the characteristic equation:

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0.$$

Theorem: If the roots of the characteristic equation are distinct real numbers, $r_1, \ldots, r_n$, then we can say:

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$

is a general solution to:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

Ex. Solve the initial value problem $y''' + 2y'' - 3y' = 0$ where

$y(0) = 2,\ \ y'(0) = -7,\ \ y''(0) = 5.$

The characteristic equation is:

$$r^3 + 2r^2 - 3r = 0$$

$$r(r^2 + 2r - 3) = 0$$

$$r(r+3)(r-1) = 0$$

$$r = 0, -3, 1.$$

So the general solution is:

$$y = c_1e^0 + c_2e^{(-3x)} + c_3e^x = c_1 + c_2e^{(-3x)} + c_3e^x$$

$$y' = -3c_2e^{(-3x)} + c_3e^x$$

$$y'' = 9c_2e^{(-3x)} + c_3e^x$$

$$2 = y(0) = c_1 + c_2e^0 + c_3e^0 = c_1 + c_2 + c_3$$

$$-7 = y'(0) = -3c_2 + c_3$$

$$5 = y''(0) = 9c_2 + c_3$$

$$-7 = -3c_2 + c_3$$

$$\underline{\ \ 5 = \ \ 9c_2 + c_3}$$

$$-12 = -12c_2$$

$$\Longrightarrow \quad c_2 = 1, \quad c_3 = -4, \quad c_1 = 5.$$

So the solution to the initial value problem is:

$$y = 5 + e^{(-3x)} - 4e^x.$$

Theorem: If the characteristic equation has a repeated root, r, of multiplicity k, then part of the general solution of the homogeneous differential equation with constant coefficients is of the form:

$$(c_1 + c_2x + c_3x^2 + \cdots + c_kx^{k-1})e^{rx}.$$

Ex. Find the general solution to $4y^{(6)} - 4y^{(5)} + y^{(4)} = 0.$

The characteristic equation is:

$$4r^6 - 4r^5 + r^4 = 0$$

$$r^4(4r^2 - 4r + 1) = 0$$

$$r^4(2r - 1)^2 = 0.$$

So $r = 0$ is a root of order 4 and $r = \frac{1}{2}$ is a double root.

General solution:

$$y = (c_1 + c_2x + c_3x^2 + c_4x^3)e^{(0)x} + (c_5 + c_6x)e^{(\frac{1}{2}x)}$$

$$y = c_1 + c_2x + c_3x^2 + c_4x^3 + (c_5 + c_6x)e^{(\frac{1}{2}x)}.$$

What happens when the roots of the characteristic equation are complex (not real) numbers? For example:

$$y'' - 2y' + 5y = 0$$

Characteristic equation: $r^2 - 2r + 5 = 0$

$$r = \frac{2 \pm \sqrt{(-2)^2 - 4(5)}}{2} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i.$$

First, when a polynomial with real coefficients has non-real roots they always come in conjugate pairs. In this case:

$$r_1 = 1 + 2i, \qquad r_2 = 1 - 2i.$$

We start with Euler's Formula: $e^{ix} = \cos x + i \sin x$.

So we have, $e^{(a+bi)x} = e^{ax+ibx} = e^{ax}e^{ibx}$

$$= e^{ax}(\cos bx + i \sin bx)$$

$$e^{(a-bi)x} = e^{ax}(\cos bx - i \sin bx).$$

If $f(x)$ and $g(x)$ are real valued functions given $F(x) = f(x) + ig(x)$, then we can define $F'(x)$ as: $F'(x) = f'(x) + ig'(x)$

Using this formula one can show that:

$$\frac{d}{dx}(e^{rx}) = re^{rx}\ , \quad \text{when } r \text{ is a complex constant.}$$

So suppose that a characteristic equation has complex roots,

$$r_1 = a + bi, \quad r_2 = a - bi.$$

Then $c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x}$ will be part of the general solution to the homogeneous equation. But we can also write:

$c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x}$

$$= c_1\big(e^{ax}(\cos bx + i \sin bx)\big) + c_2(e^{ax}(\cos bx - i \sin bx))$$

$$= (c_1 + c_2)e^{ax}(\cos bx) + i(c_1 - c_2)e^{ax}(\sin bx).$$

In general, c_1 and c_2 could be complex constants. However, we are interested in real solutions to the differential equation.

If we choose $c_1 = c_2 = \frac{1}{2}$ we get $e^{ax}(\cos bx)$ as a solution.

If we choose $c_1 = -\frac{1}{2}i, \quad c_2 = \frac{1}{2}i$ we get $e^{ax}(\sin bx)$ as a solution.

These are linearly independent so,

Theorem: If the characteristic equation has a pair of complex conjugate roots $a \pm bi$, then the corresponding part of the general solution of the homogeneous equation is:

$$e^{ax}(c_1 \cos bx + c_2 \sin bx).$$

Ex. Solve $y'' - 2y' + 5y = 0$ for which $y(0) = 3$ and $y'(0) = -5$.

Characteristic equation: $r^2 - 2r + 5 = 0$

$$r = 1 \pm 2i$$

$$a = 1, \qquad b = 2.$$

General solution:

$$y = e^x(c_1 \cos 2x + c_2 \sin 2x).$$

Particular solution:

$$y' = e^x(-2c_1 \sin 2x + 2c_2 \cos 2x) + e^x(c_1 \cos 2x + c_2 \sin 2x)$$

$$3 = y(0) = e^0(c_1 \cos 0 + c_2 \sin 0) = c_1$$

$$-5 = y'(0) = e^0(-2c_1 \sin 0 + 2c_2 \cos 0) + e^0(c_1 \cos 0 + c_2 \sin 0)$$

$$= 2c_2 + c_1$$

$$\Longrightarrow \qquad c_1 = 3, \quad c_2 = -4.$$

Particular Solution: $y = e^x(3 \cos 2x - 4 \sin 2x)$.

Ex. $3y^{(3)} - 2y'' + 12y' - 8y = 0$ has $y = e^{(\frac{2x}{3})}$ as a solution. Find the general solution.

Since $y = e^{(\frac{2x}{3})}$ is a solution, $r = \frac{2}{3}$ is a solution of the characteristic eq.:

$$3r^3 - 2r^2 + 12r - 8 = 0.$$

So $r - \frac{2}{3} = 0$ or $3r - 2 = 0$ and $3r - 2$ divides $3r^3 - 2r^2 + 12r - 8$.

$$\begin{array}{r|l} & r^2 + 4 \\ \hline 3r - 2 & 3r^3 - 2r^2 + 12r - 8 \\ & \underline{3r^3 - 2r^2} \\ & \qquad\qquad 12r - 8 \\ & \qquad\qquad \underline{12r - 8} \\ & \qquad\qquad\qquad 0. \end{array}$$

So $3r^3 - 2r^2 + 12r - 8 = (3r - 2)(r^2 + 4) = 0$, $r = \frac{2}{3}$, $r = \pm 2i$.

General solution:

$$y = c_1 e^{(\frac{2x}{3})} + e^{(0)x}(c_2 \cos 2x + c_3 \sin 2x)$$

$$y = c_1 e^{(\frac{2x}{3})} + c_2 \cos 2x + c_3 \sin 2x.$$

Repeated Complex Roots

If $a + bi$ and $a - bi$ are roots with multiplicity k then the part of the general solution corresponding to these roots are of the form:

$$\sum_{i=0}^{k-1} x^i e^{ax}(c_i \cos bx + d_i \sin bx).$$

Ex. The differential equation $y^{(4)} - 4y^{(3)} + 14y'' - 20y' + 25y = 0$ has a characteristic equation of:

$$r^4 - 4r^3 + 14r^2 - 20r + 25 = (r^2 - 2r + 5)^2 = 0.$$

Find the general solution to the differential equation.

$r^2 - 2r + 5$ has roots $r = 1 \pm 2i$, which are double roots of:

$$(r^2 - 2r + 5)^2 = 0$$

General solution:

$$y = e^x(c_1 \cos 2x + c_2 \sin 2x) + xe^x(c_3 \cos 2x + c_4 \sin 2x).$$