Higher Order Linear Differential Equations

An $\boldsymbol{n}^{\text{th}}$ **Order Linear Differential Equation** is of the form:

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_{n-1}(x)y' + P_n(x)y = F(x).$$

If $P_0(x) \neq 0$, then we can divide the above equation by $P_0(x)$:

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x).$$

The associated homogeneous equation is:

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0.$$

Theorem: Let $y_1, \dots, y_n$ be n solutions of the homogeneous linear equation on the interval I. If $c_1, \dots, c_n \in \mathbb{R}$, then the linear combination $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is also a solution to the homogeneous linear equation on the interval, I.

The proof is essentially the same as the case where $n = 2$.

Ex. $y_1(x) = e^{(-3x)}$, $y_2(x) = \cos 2x$, and $y_3(x) = \sin 2x$ are all solutions to the 3rd order homogeneous equation:

$$y^{(3)} + 3y'' + 4y' + 12y = 0.$$

Show that $y = 2e^{(-3x)} - 3\cos 2x + 2\sin 2x$ is a solution as well.

$$y = 2e^{(-3x)} - 3\cos 2x + 2\sin 2x$$

$$y' = -6e^{(-3x)} + 6\sin 2x + 4\cos 2x$$

$$y'' = +18e^{(-3x)} + 12\cos 2x - 8\sin 2x$$

$$y''' = -54e^{(-3x)} - 24\sin 2x - 16\cos 2x$$

$$\Longrightarrow \qquad y''' + 3y'' + 4y' + 12y = 0.$$

We will see later that since y_1, y_2, and y_3 are linearly independent the general solution is: $y = c_1 e^{(-3x)} + c_2\cos 2x + c_3\sin 2x.$

Theorem: Suppose $p_1, \ldots, p_n$ and f are continuous on the open interval, I, containing the point a. Then given n numbers, $b_0, b_1, \ldots, b_{n-1}$, the n^{th} order linear differential equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x)$$

has a unique solution on I with:

$$y(a) = b_0, \quad y'(a) = b_1, \quad \ldots, \quad y^{(n-1)}(a) = b_{n-1}.$$

Ex. $y = c_1 e^{(-3x)} + c_2\cos 2x + c_3\sin 2x$ is the general solution to the equation $y''' + 3y'' + 4y' + 12y = 0$. Find the unique solution where $y(0) = 4,\ y'(0) = -7,\ y''(0) = -3$.

$$y = c_1 e^{(-3x)} + c_2\cos 2x + c_3\sin 2x\,; \qquad y(0) = c_1 + c_2 = 4$$

$$y' = -3c_1 e^{-3x} - 2c_2\sin 2x + 2c_3\cos 2x\,; \quad y'(0) = -3c_1 + 2c_3 = -7$$

$$y'' = 9c_1 e^{(-3x)} - 4c_2\cos 2x - 4c_3\sin 2x\,; \quad y''(0) = 9c_1 - 4c_2 = -3$$

Multiply the first equation on the right by 4 and add it to the third equation:

$$4c_1 + 4c_2 = 16$$

$$\underline{9c_1 - 4c_2 \; = -3}$$

$$13c_1 \qquad\quad = 13$$

$$c_1 = 1$$

$$c_1 + c_2 = 4 \;\; \Rightarrow \;\; c_2 = 3$$

$$-3c_1 + 2c_3 = -7 \;\; \Rightarrow \;\; c_3 = -2.$$

So the unique solution with $y(0) = 4,\; y'(0) = -7,\; y''(0) = -3$:

$$y = e^{(-3x)} + 3\cos 2x - 2\sin 2x$$

Def. The n functions, $f_1, f_2, \dots, f_n$, are said to be **linearly dependent** on the interval, I, provided there are constants $c_1, \dots, c_n$, not all 0, such that $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$ on I.

Ex. Show that the functions $f(x) = 3,\;\; g(x) = 2 - 4x^3$, and $h(x) = x^3 - 1$ are linearly dependent on $\mathbb{R}$.

We need to show we can find c_1, c_2, c_3, not all 0, such that:

$$c_1(3) + c_2(2 - 4x^3) + c_3(x^3 - 1) = 0$$

$$3c_1 + 2c_2 - 4c_2x^3 + c_3x^3 - c_3 = 0$$

$$3c_1 - c_3 + 2c_2 + (c_3 - 4c_2)x^3 = 0$$

so $c_3 - 4c_2 = 0$ and $3c_1 - c_3 + 2c_2 = 0$.

Let $c_2 = 1$ so $c_3 = 4$

$$3c_1 - 4 + 2(1) = 0$$

$$c_1 = \frac{2}{3}$$

So: $\frac{2}{3}(3) + 1(2 - 4x^3) + 4(x^3 - 1) = 0.$

Thus $f(x), g(x)$, and $h(x)$ are linearly dependent on $\mathbb{R}$.

Def. n functions, $f_1, f_2, \ldots, f_n$, are called **linearly independent** on an interval, I, if $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$ only when $c_1 = c_2 = \cdots = c_n = 0$ for every point in I.

Ex. Show that $f(x) = x$, $g(x) = x^2$, and $h(x) = x + x^3$ are linearly independent on $\mathbb{R}$.

We need to show given any c_1, c_2, and c_3 with $c_1 f + c_2 g + c_3 h = 0$ then $c_1 = c_2 = c_3 = 0$.

$$c_1(x) + c_2(x^2) + c_3(x + x^3) = 0$$

$$(c_1 + c_3)x + c_2(x^2) + c_3(x^3) = 0$$

$\Rightarrow c_2 = 0$, $c_3 = 0$ and thus $c_1 = 0$.

Suppose $f_1, \dots, f_n$ are $(n-1)$ times differentiable on an open interval, I, then **the Wronskian** of these functions is:

$$W(f_1, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1{}' & f_2{}' & \cdots & f_n{}' \\ \vdots & \vdots & \cdots & \vdots \\ f_1{}^{(n-1)} & f_2{}^{(n-1)} & \cdots & f_n{}^{(n-1)} \end{vmatrix}$$

The Wronskian of n linearly dependent functions is 0. So to prove a set of n functions is linearly independent we just need to show that the Wronskian is non-zero (at any point in I).

Ex. Show the functions $f(x) = x,\ \ g(x) = x^2,\ \ h(x) = x + x^3$ are linearly independent on $\mathbb{R}$ by showing $W(f, g, h) \neq 0$ for some $x \in \mathbb{R}$.

$$W(f, g, h) = \begin{vmatrix} x & x^2 & x + x^3 \\ 1 & 2x & 1 + 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$$= x\begin{vmatrix} 2x & 1 + 3x^2 \\ 2 & 6x \end{vmatrix} - x^2\begin{vmatrix} 1 & 1 + 3x^2 \\ 0 & 6x \end{vmatrix} + (x + x^3)\begin{vmatrix} 1 & 2x \\ 0 & 2 \end{vmatrix}$$

$$= x\big(12x^2 - 2(1 + 3x^2)\big) - x^2(6x) + (x + x^3)(2)$$

$$= x(6x^2 - 2) - 6x^3 + 2x + 2x^3 = 2x^3 \neq 0 \text{ for } x \neq 0.$$

Since the Wronskian is not equal to 0 everywhere, f, g, h are linearly independent.

Ex. Show that $y_1(x) = x,\ y_2(x) = x^2,\ y_3(x) = x^3$, which are solutions to $x^3y''' - x^2y'' + 2xy' - 2y = 0$, are linearly independent for $x > 0$. Find the particular solution with $y(1) = 2, y'(1) = 3$, and $y''(1) = 4$.

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1{}' & y_2{}' & y_3{}' \\ y_1{}'' & y_2{}'' & y_3{}'' \end{vmatrix} = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$$= x\begin{vmatrix} 2x & 3x^2 \\ 2 & 6x \end{vmatrix} - x^2\begin{vmatrix} 1 & 3x^2 \\ 0 & 6x \end{vmatrix} + x^3\begin{vmatrix} 1 & 1+2x \\ 0 & 2 \end{vmatrix}$$

$$= x(12x^2 - 6x^2) - x^2(6x) + x^3(2) = 2x^3 \neq 0 \text{ for } x > 0.$$

$\Rightarrow y_1, y_2, y_3$ are linearly independent solutions to this differential equation.

Find the unique solution with $y(1) = 2,\ y'(1) = 3$, and $y''(1) = 4$.

$$y(x) = c_1x + c_2x^2 + c_3x^3 \qquad y(1) = c_1 + c_2 + c_3 = 2$$

$$y' = c_1 + 2c_2x + 3c_3x^2 \qquad y'(1) = c_1 + 2c_2 + 3c_3 = 3$$

$$y'' = 2c_2 + 6c_3x \qquad y''(1) = 2c_2 + 6c_3 = 4$$

$$\begin{array}{r} c_1 + 2c_2 + 3c_3 = 3 \\ \underline{c_1 + c_2 + c_3 = 2} \\ c_2 + 2c_3 = 1 \end{array} \quad \Longrightarrow \quad \begin{array}{r} 2c_2 + 4c_3 = 2 \\ \underline{2c_2 + 6c_3 = 4} \\ -2c_3 = -2 \end{array} \Longrightarrow c_3 = 1$$

$$\Rightarrow c_2 = -1,\ c_1 = 2.$$

So the unique solution is with $y(1) = 2,\ y'(1) = 3$, and $y''(1) = 4$ is:

$$y = 2x - x^2 + x^3.$$

Theorem: Suppose $y_1, \dots, y_n$ are n solutions of the homogeneous n^{th} order linear equation:

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$$

on an open interval, I, where each $p_i(x)$ is continuous.

Let $W = W(y_1, y_2, \dots, y_n)$.

a) If $y_1, y_2, \dots, y_n$ are linearly dependent, then $W = 0$ on I
b) If $y_1, y_2, \dots, y_n$ are linearly independent, then $W \neq 0$ at each point of I.

Theorem: Let $y_1, y_2, \dots, y_n$ be n linearly independent solutions of the homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$$

on an open interval, I, where $p_i(x)$ are continuous. If Y is any solution then there exist numbers $c_1, \dots, c_n$ such that

$$Y(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$$

for all x in I. $Y(x)$ is called the general solution.

Ex. We noted earlier that $y_1(x) = x,\ y_2(x) = x^2$, and $y_3(x) = x^3$ were linearly independent solutions to $x^3y''' - 3x^2y'' + 6xy' - 6y = 0$ thus, the general solution to this equation is:

$$y(x) = c_1x + c_2x^2 + c_3x^3.$$

Now let's consider the non-homogeneous n^{th} order differential equation:

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x)$$

with the associated homogeneous equation,

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0.$$

Suppose we know a single particular solution, y_p, to the non-homogeneous equation. Let Y be any solution to the non-homogeneous equation.

Let $y_c = Y - y_p$ (we call y_c a **complementary function**).

Notice that y_c is a solution of the homogeneous equation since:

$$y_c{}^{(n)} + p_1 y_c{}^{(n-1)} + \cdots + p_{n-1} y_c{}' + p_n y_c$$

$$= \left[Y^{(n)} + p_1 Y^{(n-1)} + \cdots + p_{n-1} Y' + p_n Y\right]$$

$$-[y_p{}^{(n)} + p_1 y_p{}^{(n-1)} + \cdots + p_{n-1}(x) y_p{}' + p_n y_p]$$

$$= f(x) - f(x) = 0.$$

So any solution of the non-homogeneous equation looks like:

$Y = y_c + y_p,$ where $y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$

$y_1, \dots, y_n$ are linearly independent homogenous solutions.

Ex. Given that $y_p = x^2 + 1$ is a particular solution to the equation $y'' + y' - 2y = 2x - 2x^2$, find the solution to $y'' + y' - 2y = 2x - 2x^2$ with $y(0) = 5$ and $y'(0) = 1$.

First solve the homogeneous equation to find y_c:

$$y'' + y' - 2y = 0$$

The characteristic equation is $r^2 + r - 2 = 0$

$$(r+2)(r-1) = 0$$

$$r = 1, -2 \quad \Longrightarrow \quad y_c = c_1 e^{x} + c_2 e^{-2x}.$$

So the general solution to $y'' + y' - 2y = 2x - 2x^2$ is

$$y = y_c + y_p = c_1 e^{x} + c_2 e^{-2x} + x^2 + 1.$$

$$y' = c_1 e^{x} - 2c_2 e^{-2x} + 2x$$

$$5 = y(0) = c_1 e^0 + c_2 e^0 + 0^2 + 1 = c_1 + c_2 + 1$$

$$1 = y'(0) = c_1(e^0) - 2c_2(e^0) + 2(0) = c_1 - 2c_2$$

$$4 = c_1 + c_2$$

$$1 = c_1 - 2c_2$$

$$3 = \quad 3c_2$$

$$\Rightarrow \quad c_2 = 1\,, \quad c_1 = 3.$$

So the solution to $y'' + y' - 2y = 2x - 2x^2$ with $y(0) = 5$ and $y'(0) = 1$ is:

$$y = 3e^{x} + e^{-2x} + x^2 + 1.$$