Second Order Linear Differential Equations

A **second order linear differential** equation has the form:

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

If $F(x) = 0$, we say the equation is **homogeneous**.

Ex. $x^2y'' + 2xy' + 3y = \cos x$ is a non-homogeneous linear equation.

$x^2y'' + 2xy' + 3y = 0$ is the associated homogeneous equation.

Notice that if $A(x)y'' + B(x)y' + C(x)y = F(x)$ and $A(x) \neq 0$, then we can divide the equation by $A(x)$ and get an equation of the form:

$$y'' + p(x)y' + q(x)y = f(x).$$

Here, the associated homogeneous equation is:

$$y'' + p(x)y' + q(x)y = 0.$$

Notice that if y_1 and y_2 are solutions to a homogeneous equation then so is $c_1y_1 + c_2y_2$ for any real numbers, c_1 and c_2.

If y_1, y_2 are solutions to:

$$y'' + p(x)y' + q(x)y = 0$$

then:

$$y_1'' + p(x)y_1' + q(x)y_1 = 0$$

$$y_2'' + p(x)y_2' + q(x)y_2 = 0.$$

So we have:

$$(c_1y_1 + c_2y_2)'' + p(x)(c_1y_1 + c_2y_2)' + q(x)(c_1y_1 + c_2y_2)$$
$$= (c_1y_1'' + c_2y_2'') + p(x)(c_1y_1' + c_2y_2') + q(x)(c_1y_1 + c_2y_2)$$
$$= c_1(y_1'' + p(x)y_1' + q(x)y_1) + c_2(y_2'' + p(x)y_2' + q(x)y_2)$$
$$= 0.$$

Existence and Uniqueness Theorem: Suppose $p(x), q(x)$, and $f(x)$ are continuous on an open interval, I, containing the point a. Then given any two numbers, b_0 and b_1, the equation:

$$y" + p(x)y' + q(x)y = f(x)$$

has a **unique** solution on the interval, I, that satisfies

$$y(a) = b_0, \quad y'(a) = b_1.$$

Ex. Perform the following:

a) Verify that $y_1 = e^{5x}, \quad y_2 = xe^{5x}$ are solutions to $y" - 10y' + 25y = 0$.
b) Find a unique solution for the initial conditions $y(0) = 3, \ y'(0) = 13$.

a) $y_1 = e^{5x}$ $\qquad$ $y_2 = xe^{5x}$
$y_1' = 5e^{5x}$ $\qquad$ $y_2' = 5xe^{5x} + e^{5x}$
$y_1'' = 25e^{5x}$ $\qquad$ $y_2'' = 25xe^{5x} + 5e^{5x} + 5e^{5x} = 25xe^{5x} + 10e^{5x}$

$$y_1'' - 10y_1' + 25y_1 = 25e^{5x} - 10(5e^{5x}) + 25(e^{5x})$$
$$= 25e^{5x} - 50xe^{5x} + 25e^{5x}$$
$$= 0$$

$$\begin{aligned} y_2'' - 10y_2' + 25y_2 &= 25xe^{5x} + 10xe^{5x} - 10(5xe^{5x} + e^{5x}) + 25(xe^{5x}) \\ &= 25xe^{5x} + 10xe^{5x} - 50xe^{5x} - 10e^{5x} + 25(xe^{5x}) \\ &= 0. \end{aligned}$$

b) $y = c_1y_1 + c_2y_2 = c_1e^{5x} + c_2xe^{5x}$

$3 = y(0) = c_1e^0 + c_2(0)(e^0) = c_1$

$c_1 = 3$

$y' = 5c_1e^{5x} + c_2(5xe^{5x} + e^{5x})$

$13 = y'(0) = 5c_1e^0 + c_2(0 + e^0) = 5c_1 + c_2$

$13 = 5(3) + c_2$

$c_2 = -2$

$y = 3e^{5x} - 2xe^{5x}.$

Def. Two functions on an open interval, I, are said to be **linearly Independent** on I, provided neither is a non-zero constant multiple of the other. If two functions are not linearly independent we say that they are **linearly dependent.**

Ex. $\sin x, \cos x$

$e^x,\ e^{-x}$

$\sin x,\ \ x\sin x$

$x + 2,\ x^2$

$x, |x|$

Each above pair of functions is linearly independent on $I = (-\infty, \infty)$.

Ex. $f(x) = 3(\cos^2 x - \sin^2 x)$

$g(x) = \cos 2x.$

The two functions above are not linearly independent because:

$g(x) = \cos 2x = \cos^2 x - \sin^2 x = \frac{1}{3} f(x).$

$y'' + p(x)y' + q(x)y = 0$ will always have two linearly independent solutions. Just choose two solutions where $y_1(a) = 1,\ y_1'(a) = 0$ and $y_2(a) = 0, y_2'(a) = 1.$ In fact, if we have two linearly independent solutions, y_1, y_2 of $y'' + p(x)y' + q(x)y = 0$, then every solution of this equation is of the form:

$$y = c_1 y_1 + c_2 y_2.$$

Def. Given two functions, f and g, we define the **Wronskian of f and g** as the function:

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - gf'.$$

Ex. $W(e^x, xe^x) = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix}$

$$= e^x(xe^x + e^x) - e^x(xe^x)$$

$$= e^{2x}.$$

Ex. Notice that if f and g are linearly dependent $(f = kg)$, then:

$$W(f,g) = \begin{vmatrix} kg & g \\ kg' & g' \end{vmatrix} = kgg' - kgg' = 0.$$

Ex. Show that $f(x) = e^{r_1 x}$ and $g(x) = e^{r_2 x}$ are linearly independent if $r_1 \neq r_2$.

$$f(x) = e^{r_1 x} \qquad g(x) = e^{r_2 x}$$

$$f'(x) = r_1 e^{r_1 x} \qquad g'(x) = r_2 e^{r_2 x}$$

$$W(f,g) = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix} = r_1 e^{(r_1+r_2)x} - r_2 e^{(r_1+r_2)x}$$

$$= (r_1 - r_2)e^{(r_1+r_2)x} \neq 0 \quad \text{if } r_1 \neq r_2.$$

So $f(x) = e^{r_1 x}$ and $g(x) = e^{r_2 x}$ are linearly independent if $r_1 \neq r_2$.

Theorem: If y_1 and y_2 are two solutions of the homogeneous second order linear equation $y'' + p(x)y' + q(x)y = 0$ on an open interval, I, on which p and q are continuous,

a) If y_1, y_2 are linearly dependent, then $W(y_1, y_2) = 0$ on I
b) If y_1, y_2 are linearly independent, then $W(y_1, y_2) \neq 0$ at each point of I.

Linear Second Order Equations with Constant Coefficients

Linear second order equations with constant coefficients are of the form:

$$ay'' + by' + cy = 0$$

where $a, b, c \in \mathbb{R}$.

To solve this type of equation we try $y = e^{rx}$.

Ex. Solve $y'' - 6y' + 8y = 0$.

Let $y = e^{rx}$, $y' = re^{rx}$, $y'' = r^2e^{rx}$; now substitute:

$r^2e^{rx} - 6re^{rx} + 8e^{rx} = 0$

$(r^2 - 6r + 8)e^{rx} = 0$

$e^{rx} \neq 0$ so $r^2 - 6r + 8 = 0$ (called the **characteristic**

$(r-4)(r-2) = 0$ **equation** of $y'' - 6y' + 8y = 0$)

$r = 2, 4.$

So $y_1 = e^{2x}$, $y_2 = e^{4x}$ are solutions.

If the roots of the characteristic equation are not equal (in this case $r_1 = 2, r_2 = 4, r_1 \neq r_2$), then the general solution is

$$y = c_1e^{r_1x} + c_2e^{r_2x} = c_1e^{2x} + c_2e^{4x}$$

because e^{r_1x} and e^{r_2x} are linearly independent if $r_1 \neq r_2$.

Ex. Find the general solution to $2y'' - 3y' - 2y = 0$.

First we solve the characteristic equation we get from plugging $y = e^{rx},\ \ y' = re^{rx},\ \ y'' = r^2e^{rx}$ into the differential equation:

$$2r^2e^{rx} - 3re^{rx} - 2e^{rx} = 0$$

$$(2r^2 - 3r - 2)(e^{rx}) = 0$$

$$2r^2 - 3r - 2 = 0$$

$$(2r + 1)(r - 2) = 0 \quad \Longrightarrow \quad r = -\tfrac{1}{2}\,, 2.$$

general solution: $\quad y = c_1e^{(-\frac{1}{2}x)} + c_2e^{2x}$.

Ex. Find the particular solution to $2y'' - 3y' - 2y = 0$ with $y(0) = 7$ and $y'(0) = 4$.

From the previous example we know that the general solutions is:

$$y = c_1e^{(-\frac{1}{2}x)} + c_2e^{2x}.$$

Let's use the initial conditions to find c_1 and c_2.

$$7 = y(0) = c_1 + c_2.$$

$$y'(x) = -\tfrac{1}{2}c_1e^{(-\frac{1}{2}x)} + 2c_2e^{2x}$$

$$4 = y'(0) = -\tfrac{1}{2}c_1 + 2c_2$$

So now solve the simultaneous equations:

$$7 = \quad c_1 + \; c_2$$

$$4 = -\frac{1}{2}c_1 + 2c_2.$$

Multiplying the second equation by 2 and adding we get:

$$7 = \quad c_1 + \; c_2$$

$$\underline{8 = -c_1 + 4c_2}$$

$$15 = \qquad 5c_2 \quad \Longrightarrow c_2 = 3 \;\; \Longrightarrow \quad c_1 = 4.$$

Particular solution: $\quad y = 4e^{-\frac{1}{2}x} + 3e^{2x}.$

Ex. Find the general solution to $y" - 5y' = 0$.

The characteristic equation is:

$$r^2 - 5r = 0$$

$$r(r-5) = 0$$

$$r = 0, 5.$$

general solution: $\quad y = c_1e^{(0)x} + c_2e^{5x} = c_1 + c_2e^{5x}.$

What happens if you get a double root in the characteristic equation?

Ex. Solve $y" - 4y' + 4y = 0.$

The characteristic equations is:

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

$r = 2$ is a double root.

In case of a double root, $y = xe^{rx}$ is also a solution:

$$y = xe^{2x}$$

$$y' = 2xe^{2x} + e^{2x}$$

$$y'' = 4xe^{2x} + 2e^{2x} + 2e^{2x} = 4xe^{2x} + 4e^{2x}$$

$$y" - 4y' + 4y = 0$$

So the general solution is $y(x) = (c_1 + c_2 x)e^{rx} = (c_1 + c_2 x)e^{2x}$.

Notice that $y_1 = e^{2x}$ and $y_2 = xe^{2x}$ are linearly independent because the Wronskian of these two functions is not identically zero:

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = (2x+1)e^{4x} - 2xe^{4x}$$

$$= e^{4x} \neq 0.$$