Population Models

We have already seen one model for population growth. In that model, one assumes a constant rate of growth which is proportional to the population size at time t:

$$\frac{dP}{dt} = kP(t); \;\; k > 0.$$

The general solution is:

$$P(t) = P_0 e^{kt}.$$

More generally, we could assume a birth rate, $\beta(t)$, and a death rate, $\delta(t)$, where the rate of change in the population is given by:

$$\frac{dP}{dt} = \big(\beta(t) - \delta(t)\big)P(t).$$

If $\beta(t) - \delta(t) = k$, a constant, then we get the first population model. Birth and death rates can also depend on the size of the population $P(t)$. Notice that the birth and death rates are percentages of the population. For example, suppose the population is 500 at time t and the birth rate is $0.02 = 2\%$ per year. Then over the next year there will be $.02(500) = 10$ births. So the absolute birth rate for that year is 10, but the (relative) birth rate is 0.02.

Ex. Suppose a certain lake is stocked with fish and the birth and death rates are inversely proportional to $\sqrt{P(t)}$. At the time $t = 0, P_0 = 100$ and after 6 months there are 169 fish. a) Find a formula for $P(t)$. b) How many fish are there after 1 year?

a) $\beta = \frac{a}{\sqrt{P}}$, $\delta = \frac{b}{\sqrt{P}}$, where a, b are constants.

$$\frac{dP}{dt} = (\beta - \delta)P = \left(\frac{a}{\sqrt{P}} - \frac{b}{\sqrt{P}}\right)P$$

$$\frac{dP}{dt} = (a-b)\sqrt{P}$$ Now separate variables.

$$\frac{1}{\sqrt{P}}dP = (a-b)dt$$

$$\int \frac{1}{\sqrt{P}}dP = \int (a-b)dt$$

$$2\sqrt{P} = (a-b)t + c_1$$

$$\sqrt{P} = \frac{1}{2}(a-b)t + c_2 \qquad (*)$$

$$P = (\frac{1}{2}(a-b)t + c_2)^2; \qquad let\ k = a - b$$

$$P(t) = (\frac{1}{2}kt + c_2)^2$$ Now find c_2.

$$100 = P(0) = (c_2)^2 \quad \Longrightarrow \quad c_2 = 10, \text{ since } c_2 \geq 0 \text{ from } (*).$$

$$P(t) = (\frac{1}{2}kt + 10)^2$$ Now find k.

$$169 = P(6) = (\frac{1}{2}k(6) + 10)^2$$

$$169 = (3k + 10)^2$$

$$13 = 3k + 10; \qquad 3k + 10 \neq -13, \text{ else } k < 0 \text{ and } P(t) \text{ decreases}$$

$$k = 1$$

So $P(t) = (\frac{1}{2}t + 10)^2$.

b. $P(12) = (\frac{1}{2}(12) + 10)^2 = 16^2 = 256$ fish after 1 year.

In this problem we have:

$$\frac{dP}{dt} = (a-b)\sqrt{P} = k\sqrt{P} = \sqrt{P} \qquad \text{(since } k = 1\text{)}.$$

Since $P_0 = 100$, at $t = 0$:

$$\frac{dP}{dt} = \sqrt{100} = 10\,,$$

so at $t = 0$ the absolute rate of change of the population is 10 fish per month.

But the relative rate of change (percent rate of growth/decrease) is

$$10 = \frac{dP}{dt} = \alpha P = \alpha(100)$$

$$\alpha = .1.$$

Limitations on space, food supply, and other resources could reduce the rate of population growth as the population grows. That would give us a population model that looks like:

$$\frac{dP}{dt} = \big(f(P)\big)P(t)$$

where $f(P)$ is a decreasing function of P. One simple model to use is:

$f(P) = a - bP, \;\; a, b > 0$ as the decreasing function of P.

That would give us:

$$\frac{dP}{dt} = (a - bP)P = aP - bP^2$$

where aP is the absolute birth rate and bP^2 is the absolute death rate (and a is the relative birth rate and bP is the relative death rate) , or equivalently:

$$\frac{dP}{dt} = k(M - P)P\,; \qquad k = b, \quad M = \frac{a}{b}.$$

This is called the **logistic equation**.

Ex. Solve the logistic equation, $\frac{dP}{dt} = k(M-P)P,$ when $P(0) = P_0$ and $M > P$ (population growth).

We solve this differential equation by separating variables:

$$\frac{dP}{(M-P)P} = kdt$$

$$\int \frac{dP}{(M-P)P} = \int kdt = kt + c_1.$$

Using partial fractions we get:

$$\frac{1}{(M-P)P} = \frac{A}{M-P} + \frac{B}{P} = \frac{AP+BM-BP}{(M-P)P} = \frac{(A-B)P+BM}{(M-P)P}$$

$$1 = BM \qquad \Rightarrow B = \frac{1}{M}$$

$$0 = A - B \qquad \Rightarrow A = \frac{1}{M}.$$

$$\int \frac{dP}{(M-P)P} = \frac{1}{M}\int \left(\frac{1}{M-P} + \frac{1}{P}\right)dP$$

$$= \frac{-\ln|M-P|+\ln|P|}{M} + c_2 = \frac{1}{M}\ln\left|\frac{P}{M-P}\right| + c_2.$$

Since $\int \frac{dP}{(M-P)P} = \frac{1}{M}\ln\left|\frac{P}{M-P}\right| + \mathrm{c}_2 = kt + c_1$

$$\ln\left|\frac{P}{M-P}\right| = kMt + c_3$$

$$\left|\frac{P}{M-P}\right| = e^{kMt+c_3} = e^{c_3}e^{kMt} = c_4 e^{kMt}.$$

$M > P > 0$ (population growth) so

$$\left|\frac{P}{M-P}\right| = \frac{P}{M-P}$$

$$\frac{P}{M-P} = c_4 e^{kMt}.$$

$P(0) = P_0$ so we know, $\dfrac{P_0}{M-P_0} = c_4 e^0 = c_4$

$$\frac{P}{M-P} = \frac{P_0}{M-P_0} e^{kMt}.$$

Note: If we had $0 < M < P$ (population decrease) we still get:

$$\left|\frac{P}{M-P}\right| = -\frac{P}{M-P}$$

$$-(\frac{P}{M-P}) = c_4 e^{kMt} .$$

$P(0) = P_0$ so we know, $-(\frac{P_0}{M-P_0}) = c_4 e^0 = c_4$

$$-\left(\frac{P}{M-P}\right) = -(\frac{P_0}{M-P_0})e^{kMt}$$

or $$\frac{P}{M-P} = \left(\frac{P_0}{M-P_0}\right) e^{kMt}.$$

So either way we get:

$$P = (M-P)(\frac{P_0}{M-P_0})e^{kMt}$$

$$P = \frac{M\,P_0}{M-P_0} e^{kMt} - P(\frac{P_0}{M-P_0})e^{kMt}.$$

Solving for $P(t)$ we get:

$$P(t) = \frac{M\,P_0}{P_0+(M-P_0)e^{-kMt}}\,.$$

Notice: $\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{M\,P_0}{P_0+(M-P_0)e^{-kMt}} = \frac{M\,P_0}{P_0} = M$

M is called the **carrying capacity** of the environment.

If $P_0 < M$ then $\frac{M\,P_0}{P_0+(M-P_0)e^{-kMt}} = \frac{M\,P_0}{P_0+(positive\ number)} < M$

If $P_0 > M$ then $\frac{M\,P_0}{P_0+(M-P_0)e^{-kMt}} = \frac{M\,P_0}{P_0+(negative\ number)} > M$.

Ex. Solve $\frac{dx}{dt} = 3x(5-x)$, $x(0) = 8$ (i.e., don't use the formula we derived).

$$\frac{dx}{dt} = 3x(5-x)$$

$$\frac{dx}{x(5-x)} = 3\,dt$$

$$\int \frac{dx}{x(5-x)} = \int 3\,dt = 3t + c_1$$

$$\frac{1}{x(5-x)} = \frac{A}{x} + \frac{B}{5-x}$$

$$\frac{1}{x(5-x)} = \frac{A(5-x)+Bx}{x(5-x)}$$

$$1 = A(5-x) + Bx = 5A + (B-A)x$$

$$5A = 1 \quad \Rightarrow A = \frac{1}{5}$$

$$B - A = 0 \quad \Rightarrow B = \frac{1}{5}$$

$$\frac{1}{x(5-x)} = \frac{1}{5}\left(\frac{1}{x} + \frac{1}{5-x}\right).$$

$$\int \frac{1}{5}\left(\frac{1}{x} + \frac{1}{5-x}\right) dx = \frac{1}{5}(\ln x - \ln|5 - x|) + c_2$$

$$\frac{1}{5}(\ln|x| - \ln|5 - x|) + c_2 = 3t + c_1$$

$$\frac{1}{5}\ln\left|\frac{x}{5-x}\right| = 3t + c_3$$

$$\ln\left|\frac{x}{5-x}\right| = 15t + c_4$$

$$\left|\frac{x}{5-x}\right| = e^{15t+c_4} = c_5 e^{15t}.$$

$$x(0) = 8$$

$$\left|\frac{8}{5-8}\right| = c_5 e^0 = c_5 \quad \Longrightarrow c_5 = \frac{8}{3}.$$

$\left|\frac{x}{5-x}\right| = \frac{8}{3}e^{15t}$; but $\frac{x}{5-x} < 0$, for $t > 0$, so

$$-\left(\frac{x}{5-x}\right) = \frac{8}{3}e^{15t}$$

$$\left(\frac{x}{5-x}\right) = -\frac{8}{3}e^{15t}$$

$$x = -\frac{8}{3}e^{15t}(5 - x)$$

$$x = -\frac{40}{3}e^{15t} + \frac{8}{3}xe^{15t}$$

$$x - \frac{8}{3}e^{15t}x = -\frac{40}{3}e^{15t}$$

$$x(1 - \frac{8}{3}e^{15t}) = -\frac{40}{3}e^{15t}$$

$$x = \frac{-\frac{40}{3}e^{15t}}{1-\frac{8}{3}e^{15t}} = \frac{40e^{15t}}{-3+8e^{15t}}$$

$$x(t) = \frac{40}{8-3e^{-15t}}\,.$$

Ex. Consider a population $P(t)$ of rabbits that satisfies the logistic equation: $\frac{dP}{dt} = aP - bP^2$, where we think of aP as the (absolute) birth rate and bP^2 as the (absolute) death rate. The initial population is 120 rabbits, and there are 8 births per month and 6 deaths per month at time $t = 0$. How many months does it take for $P(t)$ to reach 95% of the limiting population M?

$P_0 = 120$

initial birth rate$= 8 = aP_0 = 120a\,,\qquad \Longrightarrow \qquad a = \frac{8}{120} = \frac{1}{15}$

initial death rate$= 6 = b{P_0}^2 = (120)^2 b,\quad \Longrightarrow \quad b = \frac{6}{(120)^2} = \frac{1}{2400}$

$$\frac{dP}{dt} = aP - bP^2 = bP(\frac{a}{b} - P);$$

let $k = b = \frac{1}{2400}\ ,\ M = \frac{a}{b} = \frac{\frac{1}{15}}{\frac{1}{2400}} = 160.$

$$P(t) = \frac{M\,P_0}{P_0+(M-P_0)e^{-kMt}}$$

$$.95(M) = \frac{M\,P_0}{P_0+(M-P_0)e^{-kMt}}$$

$$.95 = \frac{P_0}{P_0+(M-P_0)e^{-kMt}} = \frac{120}{120+(160-120)e^{-(\frac{1}{2400})(160)t}}$$

$$.95 = \frac{120}{120+40e^{-\frac{t}{15}}}$$

$$(.95)\left(120 + 40e^{-\frac{t}{15}}\right) = 120$$

$$(.95)(120) + (.95)(40)e^{-\frac{t}{15}} = 120$$

$$38e^{-\frac{t}{15}} = 6$$

$$e^{-\frac{t}{15}} = \frac{6}{38} \quad \Longrightarrow \quad -\frac{t}{15} = \ln\left(\frac{6}{38}\right).$$

$t \approx 27.69$ months to reach 95% of the limiting population.