Some Substitution Methods and Exact Equations

Sometimes one can make substitutions in a differential equation that can transform the equation into one you know how to solve.

Suppose we have: $\frac{dy}{dx} = f(x, y)$.

We might be able to make a substitution $v = \alpha(x, y)$ to get an equation in v and x instead of y and x that we can solve.

Ex. Solve $\frac{dy}{dx} = (x + y + 3)^2$.

Let $v = x + y + 3$ so $y = v - x - 3$ and

$$\frac{dy}{dx} = \frac{dv}{dx} - 1.$$

Now substitute into $\frac{dy}{dx} = (x + y + 3)^2$:

$$\frac{dv}{dx} - 1 = v^2 \text{ so, } \quad \frac{dv}{dx} = 1 + v^2.$$

Now separate the variables: $\frac{dv}{1+v^2} = dx$

$$\int \frac{dv}{1+v^2} = \int dx$$

$$\tan^{-1} v + c_1 = x + c_2$$

$$\tan^{-1} v = x + c_3$$

$$v = \tan(x + c_3).$$

$$v = x + y + 3 \text{ so,}$$

$$x + y + 3 = \tan(x + c_3)$$

or

$$y = \tan(x + c_3) - x - 3.$$

So differential equations of the form $\frac{dy}{dx} = F(ax + by + c)$ can be transformed into separable equations by making the substitution $v = ax + by + c$.

Homogeneous Equations

$f(x, y)$ is a **homogeneous** function if $f(tx, ty) = f(x, y)$.

Ex. $f(x, y) = \frac{x^2+2y^2}{3x^2-5xy}$ is homogeneous since:

$$f(tx, ty) = \frac{(tx)^2+2(ty)^2}{3(tx)^2-5(tx)(ty)} = \frac{t^2(x^2+2y^2)}{t^2(3x^2-5xy)} = \frac{x^2+2y^2}{3x^2-5xy} = f(x, y).$$

Another way of saying this is that we can write $f(x, y) = F(\frac{y}{x})$.

Ex. $f(x, y) = \frac{x^2+2y^2}{3x^2-5xy} = \frac{x^2(1+2\frac{y^2}{x^2})}{x^2(3-5\left(\frac{y}{x}\right))} = \frac{(1+2\left(\frac{y}{x}\right)^2)}{3-5(\frac{y}{x})}$

So if $v = \frac{y}{x}$, $f(x, y) = F(v) = \frac{1+2v^2}{3-5v}$.

Given a differential equation in the form $\frac{dy}{dx} = f(x,y) = F(\frac{y}{x})$

make the substitution $v = \frac{y}{x}$.

So $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Substituting into $\frac{dy}{dx} = F(\frac{y}{x})$ we get:

$$v + x\frac{dv}{dx} = F(v) \quad \text{or} \quad x\frac{dv}{dx} = F(v) - v.$$

We can now separate variables:

$$\frac{1}{F(v)-v}\,dv = \frac{1}{x}\,dx.$$

Now integrate both sides of the equation.

Ex. Solve $2xy\frac{dy}{dx} = 2x^2 + 5y^2$

First put this in the form: $\frac{dy}{dx} = f(x,y)$

$$\frac{dy}{dx} = \frac{2x^2+5y^2}{2xy}.$$

Notice that $f(x,y) = \frac{2x^2+5y^2}{2xy}$ is homogenous since:

$$f(tx,ty) = \frac{2(tx)^2+5(ty)^2}{2(tx)(ty)} = \frac{2x^2+5y^2}{2xy}.$$

Now write $f(x,y) = F(\frac{y}{x})$:

$$\frac{dy}{dx} = f(x,y) = \frac{2x^2+5y^2}{2xy} = \left(\frac{x}{y}\right) + \frac{5}{2}\left(\frac{y}{x}\right).$$

Let $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$, $v = \frac{y}{x}$, $\frac{1}{v} = \frac{x}{y}$

$$v + x\frac{dv}{dx} = \frac{1}{v} + \frac{5}{2}(v)$$

$$x\frac{dv}{dx} = \frac{1}{v} + \frac{3v}{2} = \frac{3v^2+2}{2v}$$ (now separate variables)

$$\frac{2v}{3v^2+2}dv = \frac{1}{x}dx$$

$$\int \frac{2v}{3v^2+2}dv = \int \frac{1}{x}dx$$

$$\frac{1}{3}\ln(3v^2+2) + c_1 = \ln|x| + c_2$$

$$\ln(3v^2+2) = 3\ln|x| + c_3$$

$$e^{\ln(3v^2+2)} = e^{3\ln|x|+c_3} = e^{\ln|x|^3} \cdot e^{c_3}$$

$$3v^2 + 2 = c_4|x|^3; \quad c_4 = e^{c_3} > 0.$$

Substitute back $v = \frac{y}{x}$ and we get the general solution:

$$\frac{3y^2}{x^2} + 2 = c|x|^3 \quad \text{or} \quad 3y^2 + 2x^2 = c|x|^5,$$ where $c > 0$.

Ex. Solve $x^2 \frac{dy}{dx} = x^2 + xy + y^2$

$\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$; Notice $\frac{x^2+xy+y^2}{x^2} = 1 + \frac{y}{x} + \frac{y^2}{x^2}$ is homogeneous.

$$\frac{dy}{dx} = 1 + \frac{y}{x} + \frac{y^2}{x^2}.$$

Let $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$, $v = \frac{y}{x}$; so we get by substituting:

$$v + x\frac{dv}{dx} = 1 + v + v^2$$ (separate variables)

$$\frac{1}{1+v^2}\,dv = \frac{1}{x}\,dx$$

$$\int \frac{1}{1+v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\tan^{-1}(v) + c_1 = \ln|x| + c_2$$

$$\tan^{-1}(v) = \ln|x| + c_3$$

$$\tan^{-1}(\tfrac{y}{x}) = \ln|x| + c_3$$

$$\frac{y}{x} = \tan(\ln|x| + c_3)$$

$$y = x\tan(\ln|x| + c_3)$$ general solution.

Exact Differential Equations

If we have a differential equation of the form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

when is $M(x,y) + N(x,y)\frac{dy}{dx} = \frac{d}{dx}(F(x,y))$ (called **Exact** in this case)?

If we think of y as a function of x, using the chain rule the RHS becomes:

$$\frac{d}{dx}\big(F(x,y)\big) = F_x + F_y\frac{dy}{dx}.$$

So $M(x,y) + N(x,y)\frac{dy}{dx} = \frac{d}{dx}\big(F(x,y)\big) = F_x + F_y\frac{dy}{dx}$ when:

$$M(x,y) = \frac{\partial F}{\partial x}, \quad N(x,y) = \frac{\partial F}{\partial y}.$$

If that's the case, then $F(x,y) = c$ is the general solution to:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0.$$

If we are given a differential equation of the form:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

Or equivalently:

$$M(x,y)dx + N(x,y)dy = 0,$$

1) How do we know if $M(x,y) = F_x$ and $N(x,y) = F_y$ for some $F(x,y)$?

2) If we know $M(x,y) = F_x$ and $N(x,y) = F_y$ for some and $F(x,y)$, how do we find that function?

3) If we can find and $F(x,y)$ such that $M(x,y) = F_x$ and $N(x,y) = F_y$ then $F(x,y) = c$ is the general solution to:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

or equivalently:

$$M(x,y)dx + N(x,y)dy = 0.$$

Notice that if $F(x,y)$ has continuous second partial derivatives then

$F_{xy} = F_{yx}$.

So if $M(x,y) = F_x$

$N(x,y) = F_y$

then $M_y(x,y) = F_{xy}$

$N_x(x,y) = F_{yx}$.

So a necessary condition for $M(x,y) = F_x$ and $N(x,y) = F_y$ is that

$M_y(x,y) = N_x(x,y)$. So if $M_y \neq N_x$ then:

$$M(x,y) + N(x,y)\frac{dy}{dx} \neq \frac{d}{dx}\big(F(x,y)\big).$$

Theorem:

Suppose $M(x, y)$ and $N(x, y)$ have continuous partial derivatives in the open rectangle:

$$a < x < b\,, \;\; c < y < d$$

Then the differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

is exact if, and only if,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

at each point in the rectangle.

Ex. Solve $\left(x^3 + \frac{y}{x}\right) dx + (y^2 + \ln x)dy = 0, \quad x > 0.$

$$M(x, y) = x^3 + \frac{y}{x}\,, \qquad N(x, y) = y^2 + \ln x$$

$$M_y = \frac{1}{x}\,, \qquad N_x = \frac{1}{x}\,.$$

So the differential equation is exact for $x > 0$.

Now we have to find $F(x, y)$ such that:

$$\frac{\partial F}{\partial x} = x^3 + \frac{y}{x} \quad \text{and} \quad \frac{\partial F}{\partial y} = y^2 + \ln x.$$

Choose either equation and integrate with respect to the appropriate variable (x if it's F_x, y if it's F_y):

$$F(x,y) = \int F_x\, dx = \int \left(x^3 + \frac{y}{x}\right) dx = \frac{x^4}{4} + yln\, x + g(y).$$

Now differentiate $F(x,y)$ with respect to the other variable (y)

$$F_y = \ln(x) + g'(y).$$

But we know that $F_y = y^2 + \ln x$ so,

$$\ln x + g'(y) = y^2 + \ln x$$

$$g'(y) = y^2$$

$$g(y) = \frac{1}{3}y^3 + c.$$

So we now know that:

$$F(x,y) = \frac{x^4}{4} + yln\, x + g(y) = \frac{x^4}{4} + yln\, x + \frac{1}{3}y^3 + c.$$

So this means that the solution to the differential equation is:

$$F(x,y) = c$$

$$\frac{x^4}{4} + yln\, x + \frac{1}{3}y^3 = c.$$

Ex. Solve $(1+x)^2 dy - (1+y)^2 dx = 0$.

here: $M(x,y) = -(1+y)^2$, $N(x,y) = (1+x)^2$

$M_y = -2(1+y)$, $N_x = 2(1+x)$.

$M_y \neq N_x$ so the equation is not exact and we have to find another method. In this case we can separate variables.

$$(1+x)^2 dy - (1+y)^2 dx = 0$$

$$(1+x)^2 dy = (1+y)^2 dx$$

$$\frac{1}{(1+y)^2}\, dy = \frac{1}{(1+x)^2}\, dx$$

$$\int \frac{1}{(1+y)^2}\, dy = \int \frac{1}{(1+x)^2}\, dx$$

$$-\frac{1}{1+y} + c_1 = -\frac{1}{1+x} + c_2$$

$$\frac{1}{1+y} = \frac{1}{1+x} + c_3$$ now take reciprocals

$$1 + y = \frac{1}{\frac{1}{1+x} + c_3} = \frac{1}{\frac{c_3(1+x)+1}{1+x}} = \frac{1+x}{1+c_3(1+x)}$$

$$y = \frac{1+x}{1+c_3(1+x)} - 1.$$

Reducible Second Order Equations

A second order differential equation has the form

$$F(x, y, y', y") = 0.$$

For example, $xy" + y' = 4x$ and $y" = 2y(y')^3$ are second order differential equations. If either y, the dependent variable, or x, the independent variable, are missing from the second order equation (as in the cases above), then the second order equation can be reduced to a first order equation by letting $p = \frac{dy}{dx}$, where p is a function of x or y.

Ex. Solve $xy" + y' = 4x$ (missing y) by reducing it to a first order differential equation (assume $x > 0$).

Let $p(x) = y'$ so $p'(x) = y"$ and, $xp'(x) + p(x) = 4x$

Now we can solve this as follows:

$$p'(x) + \frac{1}{x}p(x) = 4.$$

This is in the form $p'(x) + R(x)p(x) = Q(x)$ so we can solve this by finding an integrating factor.

$$\rho(x) = e^{\int R(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x.$$

$$xp'(x) + p(x) = 4x$$

$$\frac{d}{dx}(xp(x)) = 4x$$

$$xp(x) = \int 4x = 2x^2 + c_1$$

$$p(x) = 2x + \frac{c_1}{x}$$

but $p(x) = \frac{dy}{dx}$ so,

$$\frac{dy}{dx} = 2x + \frac{c_1}{x}$$

$$y = x^2 + c_1 \ln x + c_2 \qquad \text{general solution.}$$

Ex. Solve $y" = 2y(y')^3$, (missing x) by reducing it.

Notice that this a non-linear differential equation because y' is raised to the 3rd power and y and y' are multiplied.

Let $p(x) = y' \qquad \Longrightarrow \qquad y" = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$

substituting into $y" = 2y(y')^3$ we get:

$$p\frac{dP}{dy} = 2y(p)^3 \qquad \text{(now separate the variables)}$$

$$\frac{1}{p^2}\frac{dp}{dy} = 2y$$

$$\frac{dp}{p^2} = 2ydy$$

$$\int \frac{dp}{P^2} = \int 2ydy$$

$$-\frac{1}{p} + c_1 = y^2 + c_2$$

$$-\frac{1}{p} = y^2 + c_3$$

$$\frac{1}{p} = -y^2 - c_3.$$

so, $p = \frac{1}{-y^2 - c_3}$.

$p = \frac{dy}{dx}$, so we get:

$$\frac{dy}{dx} = \frac{-1}{y^2+c_3}$$ separate variables again

$$(y^2 + c_3)dy = -dx$$

$$\int (y^2 + c_3)dy = \int -dx$$

$$\frac{y^3}{3} + c_3 y + c_4 = -x + c_5$$

$$x = -\frac{1}{3}y^3 - c_3 y + c_6$$ general solution.