First Order Linear Differential Equations

We will solve linear first order differential equations of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ are continuous.

Notice that we can't separate variables in this case. However, if we multiply the entire equation by $\rho(x) = e^{\int P(x)dx}$, called an **integrating factor**, something interesting happens:

$$e^{\int P(x)dx}\ \frac{dy}{dx} + P(x)e^{\int P(x)dx}(y) = Q(x)e^{\int P(x)dx}$$

Notice that since $\frac{d}{dx}(\int P(x)dx) = P(x)$ the LHS becomes:

$$\frac{d}{dx}\left[ye^{\int P(x)dx}\right] = e^{\int P(x)dx} \cdot \frac{dy}{dx} + P(x)e^{\int P(x)dx}y$$

So, $\quad \frac{d}{dx}\left[ye^{\int P(x)dx}\right] = Q(x)e^{\int P(x)dx}$

and $\quad ye^{\int P(x)dx} = \int(Q(x)e^{\int P(x)dx})dx + C$

$$y = e^{-\int P(x)dx}[\int(Q(x)e^{\int P(x)dx})dx + C].$$

Steps to solving $\frac{dy}{dx} + P(x)y = Q(x)$:

1. Calculate $\rho(x) = e^{\int P(x)dx}$
2. Multiply the differential equation by $\rho(x)$
3. Notice that $\frac{d}{dx}[\rho(x)y] = \rho(x)Q(x)$
4. Integrate both sides: $\rho(x)y = \int \rho(x)Q(x)dx + C$
5. $y = \frac{1}{\rho(x)}[\int \rho(x)Q(x)dx + C]$

Ex. Solve $2xy' + y = 10\sqrt{x}$, for $x > 0$.

Start by putting the equation in the form $\frac{dy}{dx} + P(x)y = Q(x)$:

$$y' + \frac{1}{2x}y = \frac{5}{\sqrt{x}}$$

So $P(x) = \frac{1}{2x}$, $Q(x) = \frac{5}{\sqrt{x}}$

$$\rho(x) = e^{\int P(x)dx} = e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\ln x} = (e^{\ln x})^{\frac{1}{2}} = x^{\frac{1}{2}}$$

Note: Any antiderivative of $\frac{1}{2x}$ will work and $|x| = x > 0$.

Multiply $y' + \frac{1}{2x}y = \frac{5}{\sqrt{x}}$ by $\sqrt{x}$

$$\sqrt{x}y' + \frac{1}{2\sqrt{x}}y = 5$$

Now notice $\frac{d}{dx}\left(\sqrt{x}y\right) = \sqrt{x}y' + \frac{1}{2\sqrt{x}}y$ so,

$$\frac{d}{dx}\left(\sqrt{x}y\right) = 5$$

Now integrate both sides:

$$\sqrt{x}y = \int 5dx = 5x + C$$

$$y = \frac{1}{\sqrt{x}}(5x + c) = 5\sqrt{x} + \frac{C}{\sqrt{x}}$$ general solution.

Ex. Solve the initial value problem:

$$y' = 2xy + 3x^2e^{x^2}; \quad y(0) = 5.$$

Start by putting the equation in the form $y' + P(x)y = Q(x)$.

$$y' - 2xy = 3x^2e^{x^2}$$

$$P(x) = -2x, \qquad Q(x) = 3x^2e^{x^2}$$

$$\rho(x) = e^{\int P(x)dx} = e^{\int -2xdx} = e^{-x^2}$$

$$e^{-x^2}y' - 2xe^{-x^2}y = 3x^2e^{x^2} \cdot e^{-x^2} = 3x^2$$

$$\frac{d}{dx}\left(e^{-x^2}y\right) = e^{-x^2}y' - 2xe^{-x^2}y$$

$$\frac{d}{dx}\left(e^{-x^2}y\right) = 3x^2$$

$$e^{-x^2}y = \int 3x^2\,dx = x^3 + C$$

$$y = e^{x^2}(x^3 + C)$$ general solution

$y(0) = 5$ so,

$$5 = e^{(0)^2}(0^3 + C) = C$$

$$y = e^{x^2}(x^3 + 5)$$ particular solution

Ex. Solve $y' = -y\tan x + (\cos^2 x)\sin x$, $y(0) = 3$, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

$$y' + (\tan x)y = (\cos^2 x)\sin x$$

$$P(x) = \tan x \ , \ \ Q(x) = (\cos^2 x\,)\sin x$$

$$\rho(x) = e^{\int \tan x dx} = e^{\int \frac{\sin x}{cosx} dx}$$

Let $u = \cos x$

$$du = -\sin x\, dx$$

$$-du = \sin x\, dx$$

$$\rho(x) = e^{-\int \frac{du}{u}} = e^{-\ln u} = \left(e^{\ln u}\right)^{-1} = \frac{1}{u} = \frac{1}{\cos x} = \sec x.$$

Note: $0 < cosx = u,$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, so $|u| = u$.

$$(\sec x)y' + (\sec x)(\tan x)y = (\sec x)\,(\cos^2 x)\sin x = (\cos x)\sin x$$

$$\frac{d}{dx}((\sec x)y) = (\cos x)\sin x.$$

$$(\sec x)\,y = \int (\cos x)\sin x\, dx$$

Let $u = \cos x$, $du = -\sin x\, dx$, $-du = \sin x\, dx$

$$= -\int u\, du = -\frac{u^2}{2} + C = -\frac{\cos^2 x}{2} + C$$

$$(\sec x)y = -\frac{\cos^2 x}{2} + C$$

$$y = -\frac{\cos^3 x}{2} + C(\cos x)$$ general solution.

$y(0) = 3$ so,

$$3 = -\frac{\cos^2(0)}{2} + C(\cos 0) = -\frac{1}{2} + C$$

$$\Longrightarrow \qquad C = \frac{7}{2}$$

$$y = -\frac{\cos^3 x}{2} + \frac{7}{2}\cos x$$ particular solution.