First-Order Systems

Sometimes a process is described by more than one differential equation (called a "system" of equations). For example, suppose you have two masses and two springs, one spring connected to a wall and the first mass, and one spring connecting the first and second mass. In addition, you might have an external force acting on the second mass. In this case we might want to know the position of the first mass, $x(t)$, and the position of the second mass, $y(t)$, at any time t. Notice that the force on each mass will depend on both $x(t)$ and $y(t)$. This leads to a (second order) system of differential equations of the form:

$$m_1 x'' = -k_1 x + k_2(y - x)$$

$$m_2 y'' = -k_2(y - x) + f(t)$$

Where k_1 and k_2 are spring constants and $f(t)$ is the external force acting on the second mass.

A second example of a system (in this case nonlinear) of differential equations arises when modelling a predator-prey population. In this case we imagine we have a population, $x(t)$, of a prey (e.g. rabbits) and , $y(t)$, of a predator (e.g. foxes). We might assume the following:

1. In the absence of the predator, the population the prey will grow at a rate proportional to its current population (i.e. $\frac{dx}{dt} = ax(t),\ a > 0$).
2. In the absence of the prey, the population of the predator will decline at a rate proportional to it current population (i.e., $\frac{dy}{dt} = -by(t),\ b > 0$)
3. The number of encounters is proportional to the product of the two populations.

This gives rise to a system of differential equations given by:

$$\frac{dx}{dt} = x(a - Ay)$$

$$\frac{dy}{dt} = y(-b + Bx)$$

where $a, A, b, B > 0$.

Sometimes we can solve a system of first order linear differential equations by turning them into a single differential equation of a higher order.

Ex. Solve $x'(t) = y(t)$

$$y'(t) = -x(t).$$

Notice that if $x' = y$, then by differentiating this equation we get $x'' = y'$. But we know from the second equation that $y' = -x$. Thus we have:

$$x'' = y' = -x, \quad \text{or} \quad x'' + x = 0.$$

The characteristic equation is then:

$$r^2 + 1 = 0 \quad \Rightarrow \quad r = \pm i.$$

Thus, the general solution for $x'' + x = 0$ is:

$$x(t) = Acos(t) + Bsin(t).$$

And since $y = x'(t)$, we get:

$$y = x'(t) = Bcos(t) - Asin(t).$$

So the general solution to our linear system of differential equations is a set of parametric equations:

$$x(t) = Acos(t) + Bsin(t)$$
$$y(t) = Bcos(t) - Asin(t)$$

which describes a set of curves in the plane.

In this case we can see what these curves are by:

$$\begin{aligned} x^2 + y^2 &= \left(Acos(t) + Bsin(t)\right)^2 + \left(Bcos(t) - Asin(t)\right)^2 \\ &= (A^2 + B^2)\cos^2 t + (A^2 + B^2)\sin^2 t \\ &= A^2 + B^2 \end{aligned}$$

which is a set of circles of radius $\sqrt{A^2 + B^2}$.

Ex. Solve the initial value problem:

$$x' = y; \qquad x(0) = 5$$
$$y' = 2x - y; \qquad y(0) = -1.$$

Differentiating the first equation we get:

$$x'' = y'.$$

Now using the second equation we get:

$$x'' = y' = 2x - y.$$

Using the first equation again ($y = x'$) we can now substitute into our current equation:

$$x'' = y' = 2x - y = 2x - x'.$$

Thus we have: $x'' + x' - 2x = 0.$

The characteristic equation for this differential equation is:

$$r^2 + r - 2 = 0$$

$$(r + 2)(r - 1) = 0$$

$$r = -2, 1.$$

So the general solution to $x'' + x' - 2x = 0$ is:

$$x(t) = Ae^t + Be^{-2t}.$$

We know that $y = x'$, so we can find an expression for $y(t)$ by differentiating $x(t)$.

$$y(t) = x'(t) = Ae^t - 2Be^{-2t}.$$

So our general solution to the system of differential equations is:

$$x(t) = Ae^t + Be^{-2t}$$

$$y(t) = Ae^t - 2Be^{-2t}.$$

To find A and B we will use the initial conditions: $x(0) = 5, \quad y(0) = -1.$

$$5 = x(0) = A + B$$

$$\underline{-1 = y(0) = A - 2B} \qquad \text{subtracting we get:}$$

$$6 = \qquad\qquad 3B \qquad \Rightarrow \quad B = 2,\ A = 3.$$

So the solution to this initial value system is :

$$x(t) = 3e^t + 2e^{-2t} \qquad\qquad y(t) = 3e^t - 4e^{-2t}.$$

Ex. Solve the initial value problem:

$$x' = \frac{1}{2}y \qquad x(0) = -2$$

$$y' = -8x \qquad y(0) = 16.$$

If we differentiate the first equation we get:

$$x'' = \frac{1}{2}y' .$$

From the second equation we know that $y' = -8x$ thus:

$$x'' = \frac{1}{2}y' = \frac{1}{2}(-8x) = -4x.$$

This is equivalent to: $x'' + 4x = 0.$

The characteristic equation for $x'' + 4x = 0$ is:

$$r^2 + 4 = 0$$

$$r = \pm 2i.$$

So the general solution to $x'' + 4x = 0$ is:

$$x(t) = Acos(2t) + Bsin(2t).$$

Using $x' = \frac{1}{2}y$ we can find $y(t)$:

$$x'(t) = -2Asin(2t) + 2Bcos(2t) = \frac{1}{2}y$$

$$y(t) = -4Asin(2t) + 4Bcos(2t).$$

So the general solution to the system of equations is:

$$x(t) = Acos(2t) + Bsin(2t)$$
$$y(t) = -4Asin(2t) + 4Bcos(2t).$$

Again we will use the initial conditions to find A and B.

$$-2 = x(0) = A \qquad \Rightarrow \qquad A = -2$$
$$16 = y(0) = 4B \qquad \Rightarrow \qquad B = 4.$$

Thus the solution to this initial value problem is:

$$x(t) = -2cos(2t) + 4sin(2t)$$
$$y(t) = 8sin(2t) + 16cos(2t).$$

Notice that these parametric equations describe an ellipse in the plane:

$$\left(\frac{x}{-2}\right)^2 + \left(\frac{y}{8}\right)^2 = (\cos(2t) - 2\sin(2t))^2 + (2\cos(2t) + \sin(2t))^2$$
$$\frac{x^2}{4} + \frac{y^2}{64} = \cos^2(2t) + 4\sin^2(2t) + 4\cos^2(2t) + \sin^2(2t)$$
$$\frac{x^2}{4} + \frac{y^2}{64} = 5$$
$$\frac{x^2}{20} + \frac{y^2}{320} = 1.$$

Theorem: (Existence and Uniqueness for Linear Systems)

Let $$x_1' = q_{11}(t)x_1 + q_{12}(t)x_2 + \cdots q_{1n}(t)x_n + f_1(t)$$

$$\vdots$$

$$x_n' = q_{n1}(t)x_1 + q_{n2}(t)x_2 + \cdots q_{nn}(t)x_n + f_n(t)$$

where $q_{11}, \dots, q_{nn}, f_1, \dots, f_n$ are continuous functions on an interval I containing a. Given $b_1, \dots, b_n$, real numbers, the system of differential equations has a unique solution with $x_1(a) = b_1, \dots, x_n(a) = b_n$.