

## Piecewise Continuous Functions

Recall if:

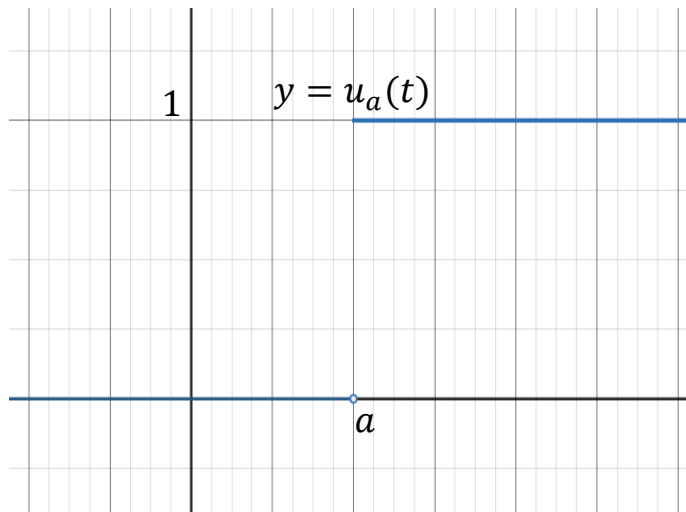
$$\begin{aligned} u(t) &= 0 \text{ if } t < 0 \\ &= 1 \text{ if } t \geq 0 \end{aligned}$$

$$\begin{aligned} u_a(t) = u(t - a) &= 0 \text{ if } t < a \\ &= 1 \text{ if } t \geq a \end{aligned}$$

Then:

$$\mathcal{L}(u(t)) = \frac{1}{s}$$

$$\mathcal{L}(u(t - a)) = \frac{e^{-as}}{s}.$$



Theorem: Translation on the  $t$ -axis.

$$\begin{aligned} \text{If } \mathcal{L}(f(t)) \text{ exists for } s > c, \text{ then } \mathcal{L}(u(t - a)f(t - a)) &= e^{-as}F(s) \\ \text{and } \mathcal{L}^{-1}(e^{-as}F(s)) &= u(t - a)f(t - a). \end{aligned}$$

Notice that:

$$\begin{aligned} u(t - a)f(t - a) &= 0 \text{ if } t < a \\ &= f(t - a) \text{ if } t \geq a. \end{aligned}$$

Proof:

$$e^{-as}F(s) = e^{-as} \int_{w=0}^{w=\infty} e^{-sw} f(w) dw$$

$$= \int_{w=0}^{w=\infty} e^{-s(w+a)} f(w) dw$$

$$\text{Let } t = w + a$$

$$dt = dw$$

$$e^{-as}F(s) = \int_{t=a}^{t=\infty} e^{-st} f(t-a) dt$$

$$= \int_{t=0}^{t=\infty} e^{-st} u(t-a) f(t-a) dt$$

$$= \mathcal{L}(u(t-a)f(t-a)).$$

Ex. Let  $F(s) = \frac{e^{-s}}{s+2}$ . Find  $\mathcal{L}^{-1}(F(s))$ .

$$\mathcal{L}^{-1}\left(e^{-as}(G(s))\right) = u(t-a)g(t-a)$$

$$\text{where } \mathcal{L}(g(t)) = G(s).$$

So for  $\mathcal{L}^{-1}\left(\frac{e^{-s}}{s+2}\right)$ ,  $a = 1$  and  $G(s) = \frac{1}{s+2}$ .

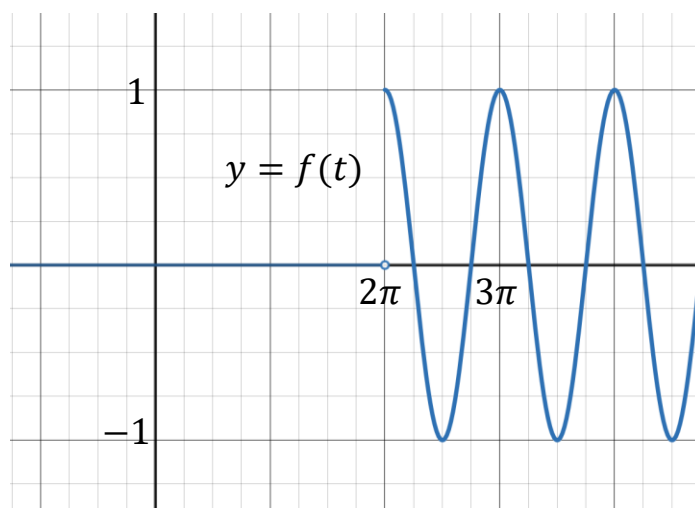
From a Laplace transform table, if  $g(t) = e^{-2t}$ , then  $G(s) = \frac{1}{s+2}$ .

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{s+2}\right) = u(t-1)g(t-1) = u(t-1)e^{-2(t-1)}.$$

Ex. Find  $\mathcal{L}(f(t))$  when

$$\begin{aligned} f(t) &= 0 && \text{if } t < 2\pi \\ &= \cos 2t && \text{if } t \geq 2\pi. \end{aligned}$$

$$\begin{aligned} f(t) &= u(t - 2\pi) \cos 2t \\ &= u(t - 2\pi) \cos(2t - 2\pi) \end{aligned}$$



$$\begin{aligned} \mathcal{L}(u(t - 2\pi) \cos(2t - 2\pi)) &= e^{-2\pi s} \mathcal{L}(\cos 2t) \\ &= e^{-2\pi s} \left( \frac{s}{s^2 + 4} \right). \end{aligned}$$

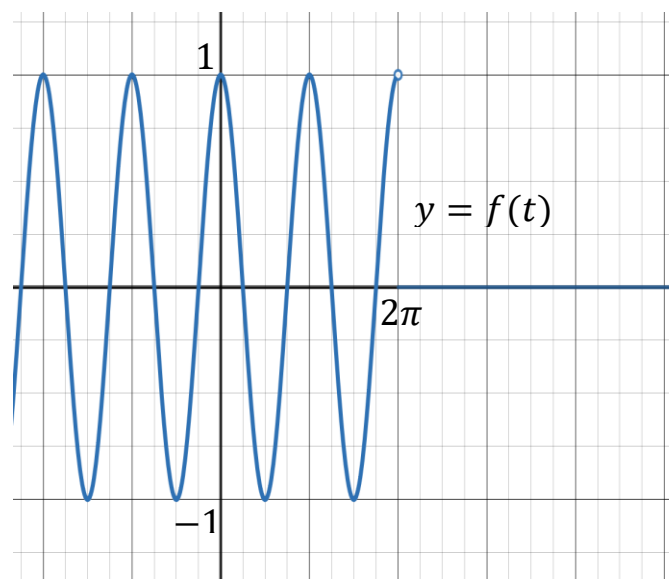
Ex. Find  $\mathcal{L}(f(t))$  when

$$\begin{aligned} f(t) &= \cos 2t && \text{if } t < 2\pi \\ &= 0 && \text{if } t \geq 2\pi. \end{aligned}$$

$$\begin{aligned} u(t - 2\pi) &= 0 && \text{if } t < 2\pi \\ &= 1 && \text{if } t \geq 2\pi. \end{aligned}$$

So

$$\begin{aligned} 1 - u(t - 2\pi) &= 1 && \text{if } t < 2\pi \\ &= 0 && \text{if } t \geq 2\pi. \end{aligned}$$



Thus,

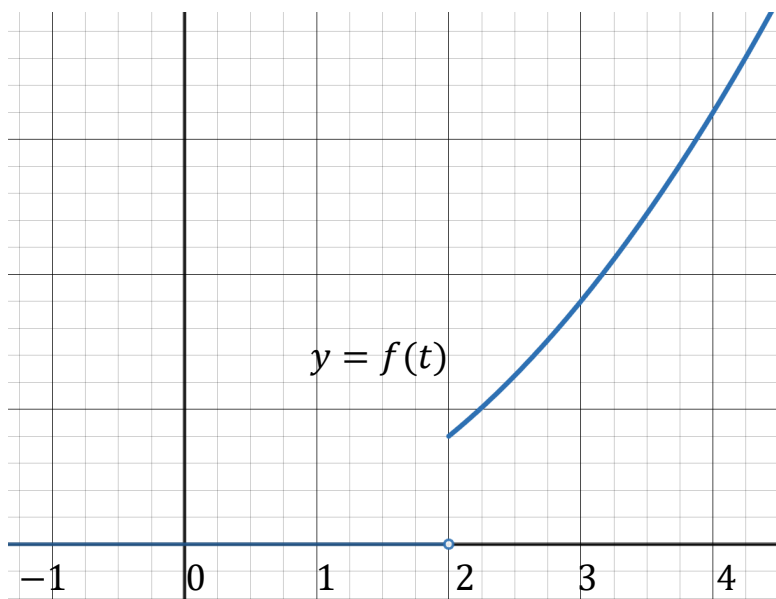
$$f(t) = (1 - u(t - 2\pi)) \cos 2t = \cos 2t - u(t - 2\pi) \cos[2(t - 2\pi)].$$

$$\begin{aligned} \mathcal{L}(\cos 2t - u(t - 2\pi) \cos[2(t - 2\pi)]) \\ = \mathcal{L}(\cos 2t) - \mathcal{L}(u(t - 2\pi) \cos[2(t - 2\pi)]) \end{aligned}$$

$$\mathcal{L}(f(t)) = \frac{s}{s^2+4} - \frac{e^{-2\pi s}}{s^2+4} = \frac{s(1-e^{-2\pi s})}{s^2+4}.$$

Ex. Find  $\mathcal{L}(g(t))$  when

$$\begin{aligned} g(t) &= 0 & \text{if } t < 2 \\ &= t^2 & \text{if } t \geq 2. \end{aligned}$$



We need to write  $g(t) = u(t - 2)f(t - 2)$ .

So after translating  $f(t)$  two units to the right we get  $t^2$ .

$$f(t) = (t + 2)^2 \text{ works (i.e. } f(t - 2) = t^2).$$

$$\begin{aligned} \mathcal{L}(g(t)) &= \mathcal{L}(u(t - 2)f(t - 2)) = e^{-2s} \mathcal{L}(t^2 + 4t + 4) \\ &= e^{-2s} \left( \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} \right). \end{aligned}$$

Ex. A mass weighing 32 lbs. is attached to the end of a spring that is stretched one foot by a force of 4 lbs (i.e.  $k = 4\text{lb/ft}$ ). Initially, The mass is at rest. At time  $t = 0$  seconds, an external force  $f(t) = 4 \cos 2t$  is applied to the mass. At time  $t = 2\pi$  seconds the force is turned off and the mass continues its motion. Find  $x(t)$ , the position of the mass at time  $t$ .

So we need to solve:  $x'' + 4x = f(t)$ ,  $x(0) = x'(0) = 0$

$$\begin{aligned} \text{where } f(t) &= 4\cos 2t & \text{if } 0 \leq t < 2\pi \\ &= 0 & \text{if } t \geq 2\pi. \end{aligned}$$

Taking the Laplace transform of this equation we get:

$$(s^2 X(s) - sx(0) - x'(0)) + 4X(s) = \frac{4s(1-e^{-2\pi s})}{s^2+4}$$

where the RHS comes from a previous example.

$$(s^2 + 4)X(s) = \frac{4s(1-e^{-2\pi s})}{s^2+4}$$

$$X(s) = \frac{4s(1-e^{-2\pi s})}{(s^2+4)^2} = \frac{4s}{(s^2+4)^2} - (e^{-2\pi s}) \frac{4s}{(s^2+4)^2}.$$

Recall that:

$$\begin{aligned}
 g(t) &= \mathcal{L}^{-1}(G(s)) = t\mathcal{L}^{-1}\left(\int_{w=s}^{\infty} G(w) dw\right) \\
 \mathcal{L}^{-1}\left(\frac{4s}{(s^2+4)^2}\right) &= t\mathcal{L}^{-1}\left(\int_{w=s}^{\infty} \frac{4w}{(w^2+4)^2} dw\right) \\
 &= t\mathcal{L}^{-1}\left(-\frac{2}{w^2+4}\Big|_{w=s}^{w=\infty}\right) \\
 &= t\mathcal{L}^{-1}\left(\left(\frac{2}{s^2+4}\right)\right) = t \sin 2t
 \end{aligned}$$

$$\mathcal{L}^{-1}(e^{-as}F(s)) = u(t-a)f(t-a)$$

$$\mathcal{L}^{-1}\left(e^{-2\pi s} \frac{s}{(s^2+4)^2}\right) = u(t-2\pi)(t-2\pi)(\sin 2(t-2\pi))$$

Since: 
$$X(s) = \frac{4s}{(s^2+4)^2} - (e^{-2\pi s}) \frac{4s}{(s^2+4)^2}.$$

we have:

$$\begin{aligned}
 x(t) &= \mathcal{L}^{-1}\left(\frac{4s}{(s^2+4)^2}\right) - \mathcal{L}^{-1}\left(e^{-2\pi s} \frac{4s}{(s^2+4)^2}\right) \\
 x(t) &= t(\sin 2t) - (u(t-2\pi))(t-2\pi)(\sin 2t) \\
 x(t) &= [t - (u(t-2\pi))(t-2\pi)] \sin 2t.
 \end{aligned}$$

In other words:

$$\begin{aligned}
 x(t) &= t \sin 2t \text{ if } t < 2\pi && \text{since } (u(t-2\pi))(t-2\pi) = 0 \\
 &= 2\pi \sin 2t \text{ if } t \geq 2\pi && \text{since } (u(t-2\pi))(t-2\pi) = t-2\pi.
 \end{aligned}$$