The Convolution Theorem/Derivatives & Integrals of Transforms

If we take the Laplace transform of $x'' + x = \cos t\,,\ x(0) = x'(0) = 0$ we get:

$$\left(s^2X(s) - sx(0) - x'(0)\right) + X(s) = \frac{s}{s^2+1}$$

$$(s^2+1)X(s) = \frac{s}{s^2+1}$$

$$X(s) = \left(\frac{s}{s^2+1}\right)\left(\frac{1}{s^2+1}\right) = \mathcal{L}(\cos t)\,\mathcal{L}(\sin t)$$

Unfortunately, $\mathcal{L}(\cos t)\,\mathcal{L}(\sin t) \neq \mathcal{L}(\cos t \cdot \sin t)$

$$= \mathcal{L}(\tfrac{1}{2}\sin 2t) = \frac{1}{s^2+4}\,.$$

However, given $H(s) = F(s)G(s)$ there is a function $h(t)$ such that,

$$\mathcal{L}\big(h(t)\big) = H(s) = F(s)G(s).$$

This function is:

$$h(t) = \int_{w=0}^{w=t} f(w)g(t-w)\,dw$$

where $\mathcal{L}\big(f(t)\big) = F(s)$ and $\mathcal{L}\big(g(t)\big) = G(s)$.

We call $h(t)$ the **convolution** of $f(t)$ and $g(t)$ and write it as:

$$h(t) = f * g(t) = \int_{w=0}^{w=t} f(w)g(t-w)dw$$

and $$\mathcal{L}(f(t) * g(t)) = \left[\mathcal{L}\big(f(t)\big)\right] \cdot \left[\mathcal{L}\big(g(t)\big)\right].$$

Ex. When we took the Laplace transform of $x'' + x = \cos t$, where $x(0) = x'(0) = 0$ we got:

$$X(s) = (\frac{s}{s^2+1})(\frac{1}{s^2+1}) = \mathcal{L}(\cos t) \cdot \mathcal{L}(\sin t).$$

Thus, $\mathcal{L}(\cos t * \sin t) = \mathcal{L}(\cos t) \cdot \mathcal{L}(\sin t)$ and $x(t) = \cos t * \sin t$ (we will calculate this shortly).

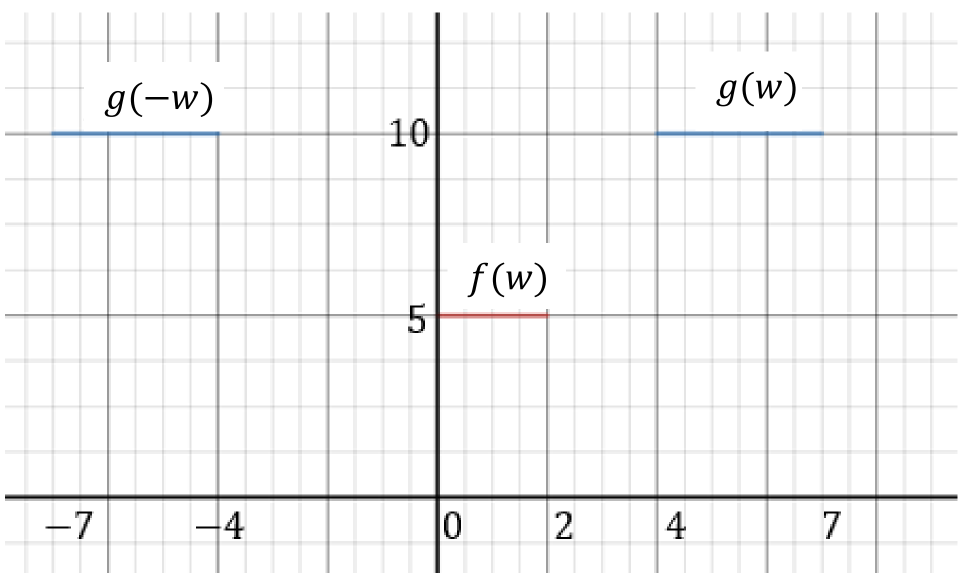
Ex. Let's calculate $(f * g)(t)$ when:

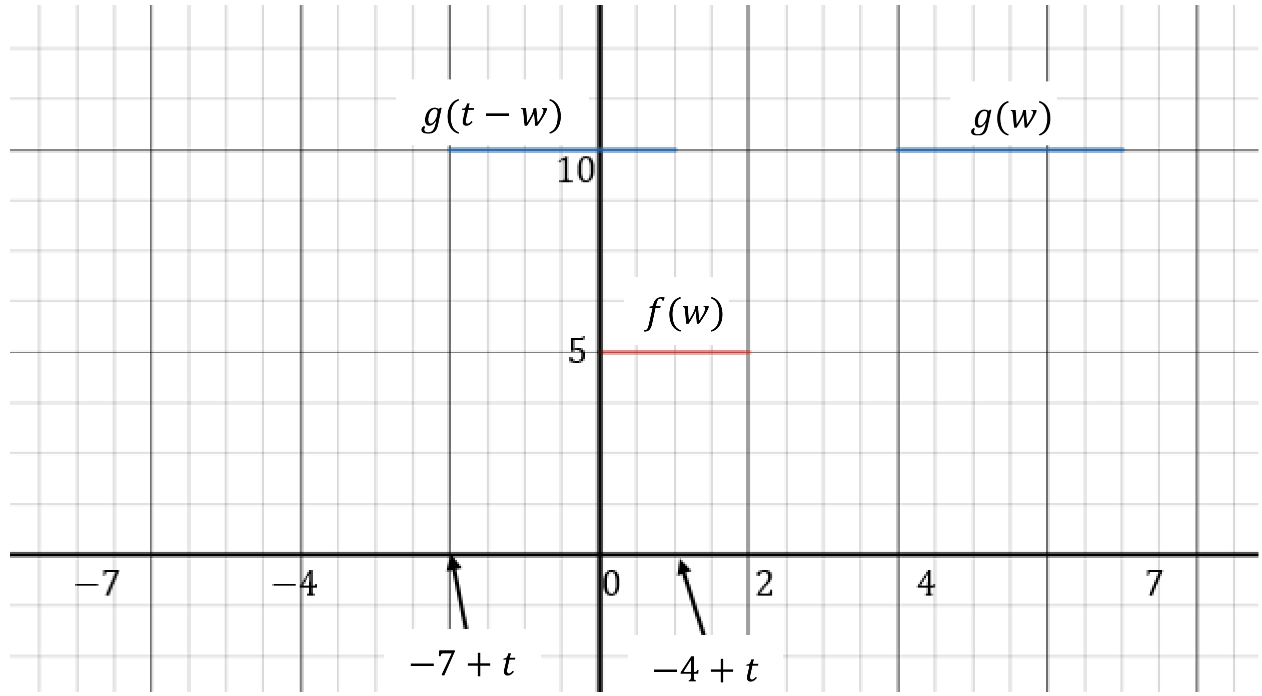
$f(t) = 5$ for $0 \leq t \leq 2$ and 0 otherwise

$g(t) = 10$ for $4 \leq t \leq 7$ and 0 otherwise.

$$(f * g)(t) = \int_{w=0}^{w=t} f(w)g(t-w)dw.$$

Let's graph $f(w), g(w), g(-w)$, and $g(t-w)$





$f(w)g(t-w) = 0$ for $0 \le t < 4$.

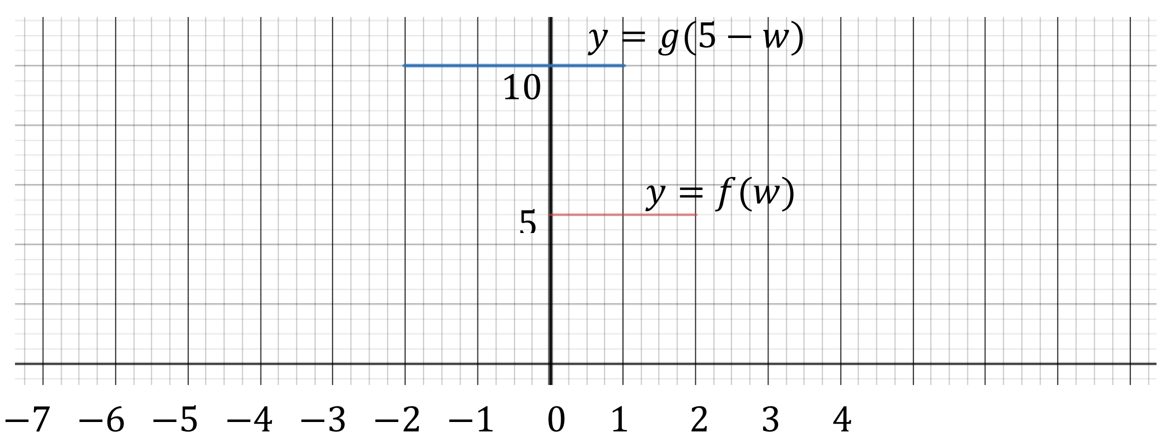
We won't get any nonzero value for the integral for $0 \le t \le 4$.

$(f * g)(4) = \int_0^4 f(w)g(4-w)dw = 0.$

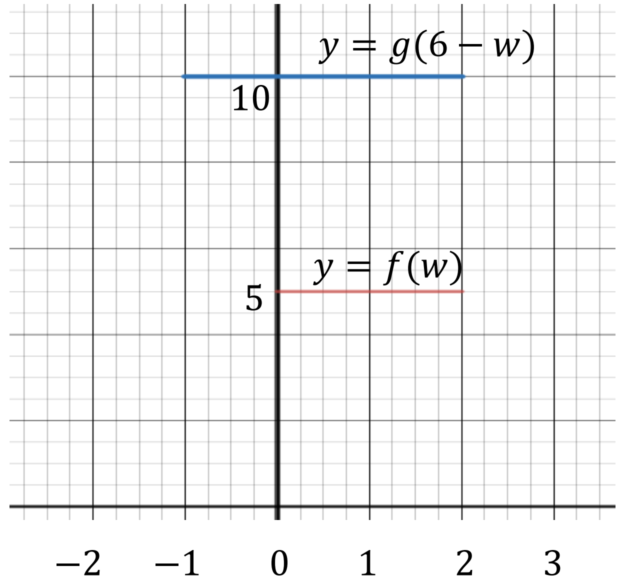
But for $4 < t < 9$, we will get nonzero values for the integral:

$(f * g)(5) = \int_0^5 f(w)g(5-w)dw$

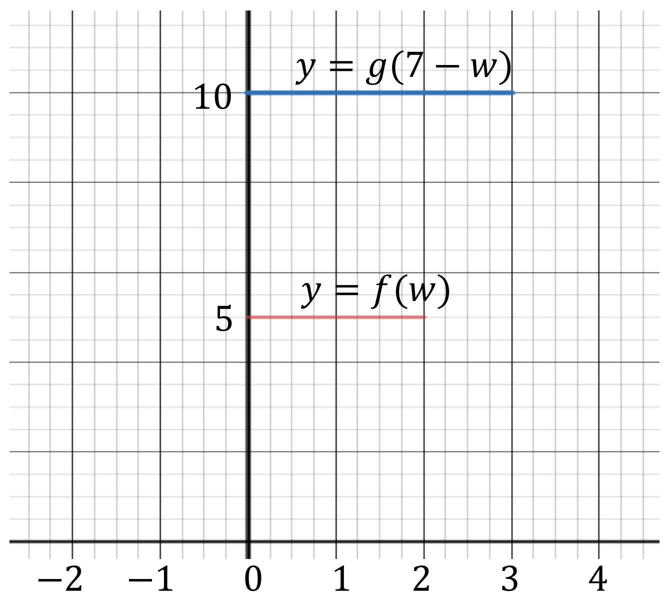
$= \int_0^1 (5)(10)dw = 50$



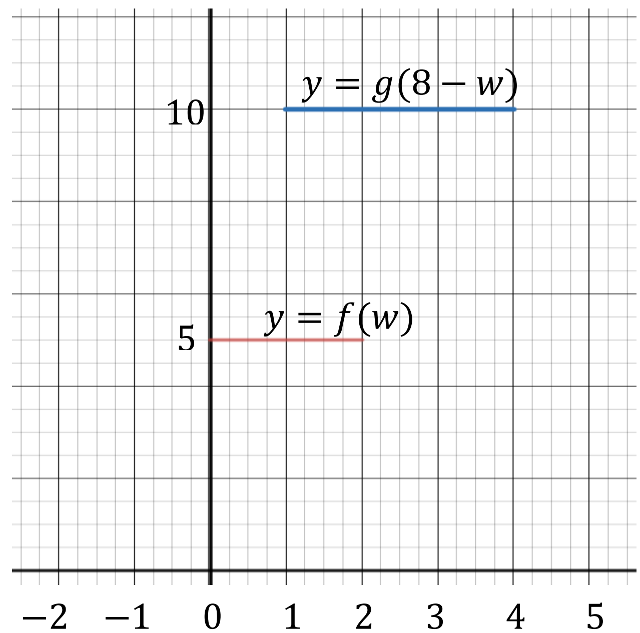
$(f * g)(6) = \int_0^2 5(10)dw = 100$



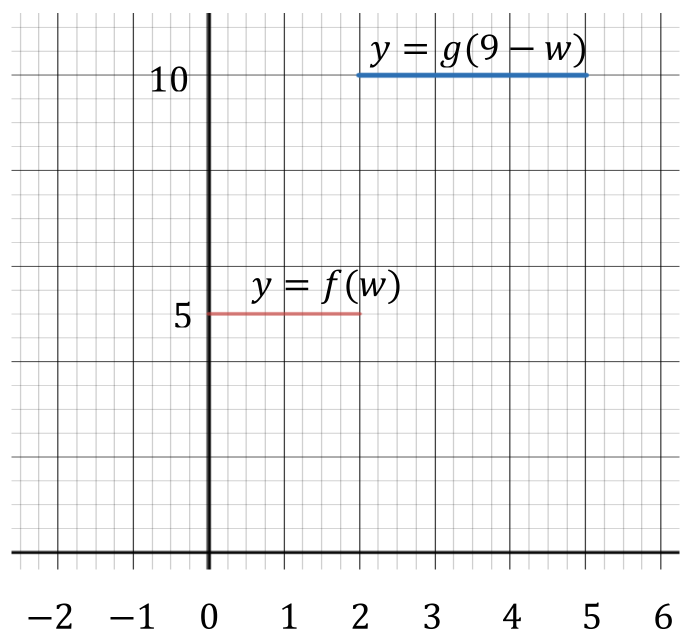
$(f * g)(7) = \int_0^2 (5)(10)dw = 100$



$$(f * g)(8) = \int_1^2 (5)(10)dw = 50$$



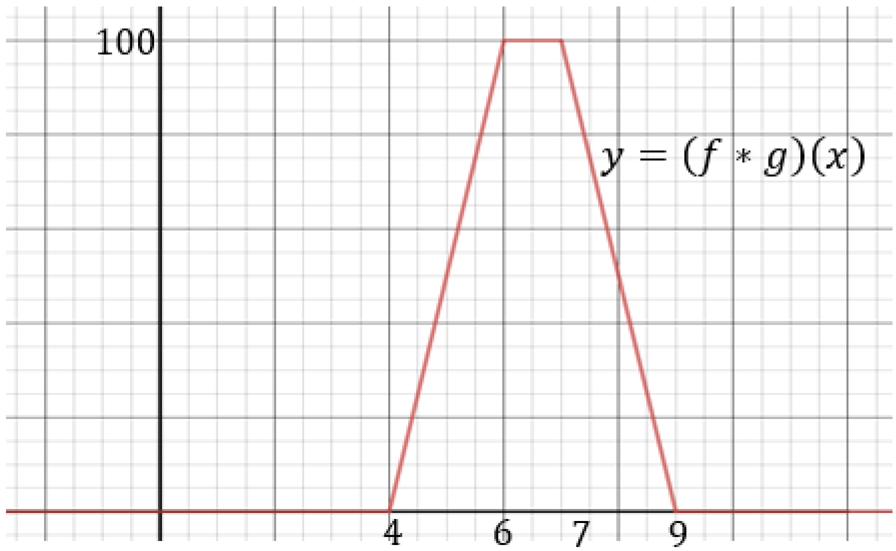
$$(f * g)(9) = \int_2^2 (5)(10)dw = 0$$



$$= \int_{w=0}^{w=t-4} 50dw = 50t - 200 \quad \text{if } 4 \le t \le 6$$

$$(f * g)(t) = \int_{w=0}^{w=2} 50dw = \quad 100 \quad \text{if } 6 < t < 7$$

$$= \int_{w=t-7}^{w=9} 50dw = 450 - 50t \quad \text{if } 7 \le t \le 9$$



Proposition: $(f * g)(t) = (g * f)(t)$.

Proof: $(f * g)(t) = \int_{w=0}^{w=t} f(w)g(t-w)dw$

Let $u = t - w$

$du = -dw$

$$= \int_{u=t}^{u=0} f(t-u)g(u)(-du)$$

$$= -\int_{u=t}^{u=0} f(t-u)g(u)\,du$$

$$= \int_{u=0}^{u=t} f(t-u)g(u)\,du$$

$$= (g * f)(t).$$

Ex. Now let's calculate $(\cos t) * (\sin t)$.

$$(\cos t) * (\sin t) = \int_{w=0}^{w=t} (\cos w)(\sin(t-w))\, dw$$

Recall: $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$.

So let $A = w, \quad B = t - w$:

$$(\cos t) * (\sin t) = \int_{w=0}^{w=t} \frac{1}{2}[\sin t - \sin(2w - t)]\, dw$$

$$= [\frac{1}{2} w \sin t + \frac{1}{4}\cos(2w - t)] \,|_{w=0}^{w=t}$$

$$= \frac{1}{2} t \sin t.$$

Thus the solution to $x'' + x = \cos t\,, \;\; x(0) = x'(0) = 0$ is

$$x(t) = \cos t * \sin t = \frac{1}{2} t \sin t\,.$$

Ex. Find $t * \sin t$.

$$t * \sin t = \int_{w=0}^{t} w \sin(t-w)\;\, dw$$ Integrate by parts.

Let $u = w \qquad v = \cos(t-w)$

$du = dw \qquad dv = \sin(t-w)dw$

$$= w\cos(t-w)\,|_{w=0}^{w=t} - \int_{w=0}^{w=t} \cos(t-w)\, dw$$

$$= (t\cos 0 - 0) + \sin(t-w)\,|_{w=0}^{w=t}$$

$$= t - \sin t.$$

Ex. Apply the convolution relationship, $\mathcal{L}(f * g) = \mathcal{L}(f) \cdot \mathcal{L}(g)$,

to find the inverse Laplace transform of $F(s) = \frac{1}{(s^2+1)^2}$.

$$F(s) = (\frac{1}{s^2+1})(\frac{1}{s^2+1})$$

$\mathcal{L}(\sin t) = \frac{1}{s^2+1}$ so

$$\frac{1}{(s^2+1)^2} = \mathcal{L}(\sin t)\, \mathcal{L}(\sin t)$$

Thus, $\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right) = x(t) = \sin t * \sin t.$

$$\sin t * \sin t = \int_{w=0}^{w=t} (\sin w)(\sin(t-w))\, dw$$

$$\sin(t-w) = \sin t \cos w - \sin w \cos t$$

$$= \int_{w=0}^{w=t} \text{sinw}(\sin t \cos w - \sin w \cos t) dw$$

$$= \int_{w=0}^{w=t} \sin t\, (\sin w)(\cos w) - \cos t\, (\sin^2 w)\, dw$$

$$= \int_{w=0}^{w=t} \sin t\, (\sin w)(\cos w)\, dw - \int_{w=0}^{w=t} \cos t\, (\sin^2 w)\, dw$$

Let $u = \sin w$, $du = \cos w \ dw$; $\qquad \sin^2 w = \frac{1}{2} - \frac{1}{2}\cos 2w$

$$= \int_{u=0}^{u=\sin(t)} (\sin t)u \, du - \int_{w=0}^{w=t} \cos t \left(\frac{1}{2} - \frac{1}{2}\cos 2w\right) \, dw$$

$$= (\sin t)\frac{u^2}{2} |_{u=0}^{u=\sin t} - [\frac{1}{2} w \cos t - (\frac{1}{4}\cos t) \sin 2w |_{w=0}^{w=t}$$

$$= \frac{\sin^3 t}{2} - \left[\frac{1}{2} t \cos t - \frac{1}{4}\cos t \sin 2t\right].$$

$$x(t) = \frac{\sin^3 t}{2} - \frac{1}{2}\text{t}\cos t + \frac{1}{4}(\cos t)(\sin 2t)$$
$$= \frac{1}{2}\sin^3 t - \frac{1}{2} tcost + \frac{1}{4}(cost)(2sint(cost))$$
$$= \frac{1}{2} sint - \frac{1}{2} tcost.$$

Note: $\sin t * \sin t$ can also be calculated by using:

$$(sinA)(sinB) = \frac{1}{2}[\cos(A - B) - \cos(A + B)].$$

Differentiation of Transforms

If $\mathcal{L}(t\,f(t))$ exists, then $\mathcal{L}\big(t\,f(t)\big) = -F'(s)$, or equivalently:

$$f(t) = \mathcal{L}^{-1}[F(s)] = -\frac{1}{t}\mathcal{L}^{-1}[F'(s)].$$

(We'll prove the above at the end of the section).

We also have:

$$\mathcal{L}\big(t^n f(t)\big) = (-1)^n F^{(n)}(s).$$

Ex. Find $\mathcal{L}(t^2 \cos kt)$.

$$\mathcal{L}(t^2 \cos kt) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+k^2}\right)$$

$$= \frac{d}{ds}\left(\frac{\left(s^2+k^2\right)-s(2s)}{(s^2+k^2)^2}\right) = \frac{d}{ds}\frac{\left(-s^2+k^2\right)}{(s^2+k^2)^2}$$

$$= \left[\frac{(s^2+k^2)^2(-2s)-(-s^2+k^2)2\left(s^2+k^2\right)(2s)}{(s^2+k^2)^4}\right]$$

$$= \left[\frac{-2s\left(s^2+k^2\right)-4s(-s^2+k^2)}{(s^2+k^2)^3}\right] = \frac{2s^3-6k^2s}{(s^2+k^2)^3}\,.$$

Ex. Find $\mathcal{L}^{-1}(\ln[(s^2+1)(s^2+9)])$.

$$F(s) = \ln[(s^2+1)(s^2+9)] = \ln(s^2+1) + \ln(s^2+9)$$

We know: $\mathcal{L}^{-1}[F(s)] = -\frac{1}{t}\mathcal{L}^{-1}[F'(s)]$

$$F'(s) = \frac{2s}{s^2+1} + \frac{2s}{s^2+9}$$

$$\mathcal{L}^{-1}(\ln[(s^2+1)(s^2+9)]) = -\frac{1}{t}\left[\mathcal{L}^{-1}\left(\frac{2s}{s^2+1}\right) + \mathcal{L}^{-1}\left(\frac{2s}{s^2+9}\right)\right]$$

Recall: $\mathcal{L}(\cos kt) = \frac{s}{s^2+k^2}$.

$$\mathcal{L}^{-1}\big(F(s)\big) = -\frac{1}{t}[2\cos t + 2\cos 3t].$$

Ex. Transform the following equation and find a nontrivial solution with $x(0) = 0$.

$$tx'' + (3t-1)x' + 3x = 0.$$

$$\mathcal{L}(tx'' + (3t-1)x' + 3x) = \mathcal{L}(tx'') + 3\mathcal{L}(tx') - \mathcal{L}(x') + 3\mathcal{L}(x)$$

$$\mathcal{L}(x'') = s^2X(s) - x(0)s - x'(0) = s^2X(s) - x'(0)$$

$$\mathcal{L}(tx'') = -\frac{d}{ds}\big(s^2X(s) - x'(0)\big) = -[s^2X'(s) + 2sX(s)]$$

$$\mathcal{L}(x') = sX(s) - x(0) = sX(s)$$

$$\mathcal{L}(tx') = -\frac{d}{ds}\big(sX(s)\big) = -sX'(s) - X(s)$$

$$\mathcal{L}(tx'') + 3\mathcal{L}(tx') - \mathcal{L}(x') + 3\mathcal{L}(x)$$
$$= [-s^2X'(s) - 2sX(s)] + 3[-sX'(s) - X(s)] - sX(s) + 3X(s) = 0$$

$$\implies (-s^2 - 3s)X'(s) - 3sX(s) = 0.$$

So we need to solve this differential equation for $X(s)$:

$$(-s^2 - 3s)X'(s) = 3sX(s)$$

$$\frac{X'(s)}{X(s)} = \frac{-3s}{s^2+3s} = \frac{-3s}{s(s+3)} = -\frac{3}{s+3}$$

Integrating both sides we get:

$$\ln X(s) = -3\ln|s+3| + C \quad \implies \quad X(s) = e^{-3\ln|s+3|+C} = \frac{C'}{(s+3)^3}.$$

Recall: $\mathcal{L}(e^{at}t^n) = \frac{n!}{(s-a)^{n+1}}$

$$\implies x(t) = C''t^2e^{-3t}.$$

Integration of Transforms

Theorem: If $\lim_{t\to 0^+} \frac{f(t)}{t}$ exists and is finite and $f(t)$ is piecewise continuous with $|f(t)| \le Ke^{at}$; for constants K and a, as $t \to \infty$, then

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_{w=s}^{\infty} F(w)\, dw$$

or

$$f(t) = \mathcal{L}^{-1}(F(s)) = t\mathcal{L}^{-1}[\int_s^{\infty} F(w)\, dw].$$

Ex. Find $\mathcal{L}\left(\frac{\sin t}{t}\right)$.

$$\mathcal{L}\left(\frac{\sin t}{t}\right) = \int_{w=s}^{\infty} \mathcal{L}(\sin t)\, dw = \int_{w=s}^{\infty} \frac{1}{w^2+1}\, dw$$

$$= \tan^{-1} w \,|_s^{\infty} = \frac{\pi}{2} - \tan^{-1} s.$$

Ex. Find $\mathcal{L}^{-1}(\frac{s}{(s^2+1)^2})$.

$$\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right) = t\mathcal{L}^{-1}[\int_s^{\infty} \frac{w}{(w^2+1)^2}\, dw]$$

$$= t\mathcal{L}^{-1}[-\frac{1}{2}\left(\frac{1}{w^2+1}\right) |_s^{\infty}]$$

$$= t\mathcal{L}^{-1}\left[\frac{1}{2}\left(\frac{1}{s^2+1}\right)\right] = \frac{1}{2} t sin(t).$$

Proof of $\mathcal{L}\big(t\, f(t)\big) = -F'(s)$:

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty \frac{d}{ds} (e^{-st} f(t))\, dt$$

$$= \int_0^\infty -t f(t) e^{-st}\, dt$$

$$= -\mathcal{L}\big(t\, f(t)\big)$$

$$\Longrightarrow \quad \mathcal{L}\big(t\, f(t)\big) = -F'(s).$$