Translation of Laplace Transforms and Partial Fractions

Suppose $R(s) = \frac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ are polynomials, and the degree of $P(s)$ is less than $Q(s)$.

The portion of the partial fraction decomposition of $R(s)$ corresponding to a linear factor $(s-a)$ of multiplicity n is a sum of terms of the form:

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_n}{(s-a)^n}$$

where $A_1, A_2, \ldots A_n$ are constants.

Ex. Find the form of the partial fraction expansion of

$$R(s) = \frac{1}{(s^2-1)(s+1)}.$$

$$\frac{1}{(s^2-1)(s+1)} = \frac{1}{(s-1)(s+1)(s+1)} = \frac{1}{(s-1)(s+1)^2}$$

$$\frac{1}{(s-1)(s+1)^2} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{(s+1)^2}.$$

The portion of the partial fraction decomposition corresponding to the irreducible quadratic factor $(s-a)^2 + b^2$ of multiplicity n is a sum if n partial fractions of the form:

$$\frac{A_1 s + B_1}{(s-a)^2+b^2} + \frac{A_2 s + B_2}{((s-a)^2+b^2)^2} + \cdots + \frac{A_n s + B_n}{((s-a)^2+b^2)^n}.$$

Ex. Find the form of the partial fraction expansion for

$$R(s) = \frac{1}{(s-1)^2(s^2+1)^2}.$$

$$\frac{1}{(s-1)^2(s^2+1)^2} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+D}{s^2+1} + \frac{Es+F}{(s^2+1)^2}.$$

Note: any quadratic polynomial with real coefficients can always put in the form of $(s-a)^2 \pm b^2$ by completing the square.

Ex. Put $s^2 + 8s + 40$ in the form $(s-a)^2 \pm b^2$ by completing the square.

$$\begin{aligned} s^2 + 8s + 40 &= s^2 + 8s + 16 - 16 + 40 \\ &= (s+4)^2 + 24. \end{aligned}$$

Theorem: If $F(s) = \mathcal{L}(f(t))$ exists for $s > c$, then $\mathcal{L}(e^{at} f(t))$ exists for $s > (a+c)$ and $\mathcal{L}\big(e^{at} f(t)\big) = F(s-a)$, or equivalently, $\mathcal{L}^{-1}\big(F(s-a)\big) = e^{at} f(t)$. Thus, the translation $s \to (s-a)$ in the transform corresponds to multiplication of the original function by e^{at}.

Proof:

$$\mathcal{L}\big(e^{at} f(t)\big) = \int_0^\infty e^{-st} e^{at} f(t)dt = \int_0^\infty e^{-(s-a)t} f(t)dt = F(s-a).$$

Below is a table relating $\mathcal{L}(f(t))$ to $\mathcal{L}\left(e^{at}f(t)\right)$ for a few functions, $n \in \mathbb{Z}^+$.

$f(t)$	$\mathcal{L}(f(t))$	$e^{at}f(t)$	$\mathcal{L}(e^{at}f(t))$
t^n	$\frac{n!}{s^{n+1}}$	$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$
$\cos kt$	$\frac{s}{s^2+k^2}$	$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$\sin kt$	$\frac{k}{s^2+k^2}$	$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$

Ex. Find $\mathcal{L}(t^3e^{-7t})$.

In this case $n = 3$ and $a = -7$ in the table above.

So $\mathcal{L}(t^3e^{-7t}) = \frac{3!}{(s+7)^4} = \frac{6}{(s+7)^4}$.

Ex. Find $\mathcal{L}\left(e^{\frac{t}{2}}\, cos2t\right)$.

In this case $a = \frac{1}{2},\ k = 2.$ So $\mathcal{L}\left(e^{\frac{t}{2}}\, cos2t\right) = \frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+4}$.

Ex. Find the inverse Laplace transform of $F(s) = \frac{s+5}{s^2+4s+5}$.

First complete the square in the denominator (since we can't factor it).

$$\frac{s+5}{s^2+4s+5} = \frac{s+5}{s^2+4s+4-4+5} = \frac{s+5}{(s+2)^2+1}.$$

Now write the numerator in terms of $s+2$ so we can use our table to find $\mathcal{L}^{-1}$.

$$\frac{s+5}{s^2+4s+5} = \frac{(s+2)+3}{(s+2)^2+1} = \frac{(s+2)}{(s+2)^2+1} + \frac{3}{(s+2)^2+1}.$$

$$\mathcal{L}^{-1}\left(\frac{s+5}{s^2+4s+5}\right) = \mathcal{L}^{-1}\left(\frac{s+2}{(s+2)^2+1}\right) + \mathcal{L}^{-1}\left(\frac{3}{(s+2)^2+1}\right)$$

$$= e^{-2t}\cos(t) + 3e^{-2t}\sin(t).$$

Ex. Consider a mass, spring, dashpot system with $m = 1$, $k = 125$, and, $c = 10$. Suppose $x(0) = 6$ and $x'(0) = 50$. Find $x(t)$.

$$x'' + 10x' + 125x = 0; \qquad x(0) = 6, \; x'(0) = 50$$

Now take the Laplace transform of the differential equation and then solve for $X(s)$

$$\mathcal{L}(x'') + 10\mathcal{L}(x') + 125\mathcal{L}(x) = 0$$

$$[s^2X(s) - sx(0) - x'(0)] + 10[sX(s) - x(0)] + 125X(s) = 0$$

$$(s^2 + 10s + 125)X(s) - 6s - 50 - 60 = 0$$

$$(s^2 + 10s + 125)X(s) = 6s + 110$$

$$X(s) = \frac{6s+110}{s^2+10s+125}.$$

Now complete the square for the expression for $X(s)$.

$$X(s) = \frac{6s+110}{s^2+10s+25+100}$$

$$= \frac{6s+110}{(s+5)^2+100} \quad \text{write the numerator in terms of } s+5$$

$$= \frac{6(s+5)+80}{(s+5)^2+10^2}\,.$$

$$X(s) = 6\left[\frac{s+5}{(s+5)^2+10^2}\right] + 8\left[\frac{10}{(s+5)^2+10^2}\right].$$

Now find $\mathcal{L}^{-1}$.

Notice that:

$$\mathcal{L}(e^{at}\cos(kt)) = \frac{s-a}{(s-a)^2+k^2}$$

$$\mathcal{L}(e^{at}\sin(kt)) = \frac{k}{(s-a)^2+k^2}\,.$$

So we have:

$$x(t) = \mathcal{L}^{-1}\big(X(s)\big) = 6\mathcal{L}^{-1}\left(\frac{s+5}{(s+5)^2+10^2}\right) + 8\mathcal{L}^{-1}\left(\frac{10}{(s+5)^2+10^2}\right)$$

$$x(t) = e^{-5t}(6\cos 10t + 8\sin 10t).$$

Ex. Solve the initial value problem $x'' + 4x' + 13x = 13$, with

$x(0) = x'(0) = 0.$

$$\mathcal{L}(x'') + 4\mathcal{L}(x') + 13\mathcal{L}(x) = \mathcal{L}(13)$$

$$[s^2X(s) - sx(0) - x'(0)] + 4[sX(s) - x(0)] + 13X(s) = \frac{13}{s}$$

$$(s^2 + 4s + 13)X(s) = \frac{13}{s}$$

$$X(s) = \frac{13}{s(s^2+4s+13)}.$$

Now write $\frac{13}{s(s^2+4s+13)}$ using partial fractions.

$$\frac{13}{s(s^2+4s+13)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+13}$$

$$= \frac{A(s^2+4s+13)+(Bs+c)(s)}{s(s^2+4s+13)}$$

So, $13 = A(s^2 + 4s + 13) + (Bs + C)(s)$

$= (A + B)s^2 + (4A + C)s + 13A.$

$\Longrightarrow \; A + B = 0, \;\; 4A + C = 0, \;\; 13A = 13.$

Solving we get: $A = 1, \;\; B = -1, \;\; C = -4.$

Thus we have: $X(s) = \frac{13}{s(s^2+4s+13)} = \frac{1}{s} - \frac{s+4}{s^2+4s+13}.$

Now complete the square on $s^2 + 4s + 13$.

$$\frac{s+4}{s^2+4s+13} = \frac{s+4}{s^2+4s+4-4+13}$$

$$= \frac{s+4}{(s+2)^2+9} \qquad \text{write the numerator in terms of } s + 2.$$

$$= \frac{s+2}{(s+2)^2+9} + \frac{2}{(s+2)^2+9}$$

$$= \frac{s+2}{(s+2)^2+9} + \frac{2}{3}\left(\frac{3}{(s+2)^2+9}\right)$$

Note: We factored $\frac{2}{3}$ from the second term above to make what's in the parentheses look like the Laplace transform of $e^{at}\sin(kt)$.

Thus we have:

$$X(s) = \frac{1}{s} - \frac{s+2}{(s+2)^2+9} - \frac{2}{3}\left(\frac{3}{(s+2)^2+9}\right).$$

Notice that:

$$\mathcal{L}(1) = \frac{1}{s}$$

$$\mathcal{L}(e^{at}\cos(kt)) = \frac{s-a}{(s-a)^2+k^2}$$

$$\mathcal{L}(e^{at}\sin(kt)) = \frac{k}{(s-a)^2+k^2}.$$

$$x(t) = \mathcal{L}^{-1}(X(s))$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{s+2}{(s+2)^2+9}\right) - \frac{2}{3}\mathcal{L}^{-1}\left(\frac{3}{(s+2)^2+9}\right)$$

$$= 1 - e^{-2t}\cos(3t) - \frac{2}{3}e^{-2t}\sin(3t).$$