Solving Initial Value Problems with Laplace Transforms

We will solve differential equations with constant coefficients using Laplace transforms by transforming the differential equation.

$$ax''(t) + bx'(t) + cx(t) = f(t)$$

$$a\mathcal{L}\big(x''(t)\big) + b\mathcal{L}\big(x'(t)\big) + c\mathcal{L}\big(x(t)\big) = \mathcal{L}\big(f(t)\big).$$

To do this we need the following:

Theorem: Suppose $f(t)$ is continuous and piecewise differentiable for $t \geq 0$

and there exist non-negative constants M, c, and T such that

$|f(t)| \leq Me^{ct}$ for $t \geq T$ then

$$\mathcal{L}\big(f'(t)\big) = s\mathcal{L}\big(f(t)\big) - f(0\ )$$

$$= sF(s) - f(0); \quad s > c.$$

Proof: Using integration by parts we get:

$$\mathcal{L}\big(f'(t)\big) = \int_0^\infty e^{-st} f'(t)dt = e^{-st} f(t)\big|_0^\infty + s\int_0^\infty e^{-st}\ f(t)dt$$

Let $u = e^{-st}$ $\qquad v = f(t)$

$du = -se^{-st}dt \quad dv = f'(t)dt$

$\mathcal{L}\big(f'(t)\big) = -f(0) + s\mathcal{L}\big(f(t)\big)$ (since $\lim_{t\to\infty} e^{-st} f(t) = 0$, for $s > c$)

$\mathcal{L}\big(f'(t)\big) = sF(s) - f(0).$

Now to find $\mathcal{L}\big(f''(t)\big)$ we assume $g(t) = f'(t)$ and use the previous relationship since $f''(t) = g'(t)$:

$$\mathcal{L}\big(f''(t)\big) = \mathcal{L}\big(g'(t)\big) = s\mathcal{L}\big(g(t)\big) - g(0)$$

$$= s\mathcal{L}\big(f'(t)\big) - f'(0)$$

$$= s\big[s\mathcal{L}\big(f(t)\big) - f(0)\big] - f'(0)$$

$$\mathcal{L}\big(f''(t)\big) = s^2F(s) - sf(0) - f'(0).$$

Similarly:

$$\begin{aligned}\mathcal{L}\big(f'''(t)\big) &= s\mathcal{L}\big(f''(t)\big) - f''(0)\\ &= s^3F(s) - s^2f(0) - sf'(0) - f''(0)\end{aligned}$$

and

$$\mathcal{L}\left(f^{(n)}(t)\right) = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) -$$
$$\ldots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Ex. Solve the initial value problem $x'' - x' - 2x = 0, \quad x(0) = 7,\ x'(0) = 2.$

$$\mathcal{L}\big(x''(t)\big) - \mathcal{L}\big(x'(t)\big) - 2\mathcal{L}(x) = 0$$

$\mathcal{L}\big(x''(t)\big) = s^2X(s) - s\big(x(0)\big) - x'(0)$, where $X(s) = \mathcal{L}\big(x(t)\big)$

$$\mathcal{L}\big(x'(t)\big) = sX(s) - x(0)$$

Substituting into the transformed equation:

$$\big[s^2X(s) - s\big(x(0)\big) - x'(0)\big] - [sX(s) - x(0)] - 2X(s) = 0$$

$$(s^2X(s) - 7s - 2) - (sX(s) - 7) - 2X(s) = 0$$

$$X(s)(s^2 - s - 2) - 7s + 5 = 0.$$

Solving for $X(s)$, we get:

$$X(s) = \frac{7s-5}{s^2-s-2} = \frac{7s-5}{(s-2)(s+1)}.$$

Now use partial fractions:

$$\frac{7s-5}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

$$= \frac{A(s+1)+B(s-2)}{(s-2)(s+1)}.$$

$7s - 5 = A(s+1) + B(s-2)$ (can also solve for A, B by

$= (A+B)s + (A-2B)$ letting $s = -1$, then $s = 2$)

$$7 = A + B$$

$$-5 = A - 2B$$

$$\Longrightarrow \quad A = 3, \quad B = 4.$$

Now we can write:

$$X(s) = \frac{7s-5}{(s-2)(s+1)} = \frac{3}{s-2} + \frac{4}{s+1}$$

The solution, $x(t)$, is the inverse Laplace transform of $X(s)$

$$x(t) = \mathcal{L}^{-1}\big(X(s)\big) = \mathcal{L}^{-1}\left(\frac{3}{s-2} + \frac{4}{s+1}\right)$$

$$= 3\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) + 4\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$x(t) = 3e^{2t} + 4e^{-t}.$$

Ex. Solve the initial value problem:

$$x'' + x = \cos 3t\,, \qquad x(0) = 1, \quad x'(0) = 0.$$

$$\mathcal{L}(x'') + \mathcal{L}(x) = \mathcal{L}(\cos 3t)$$

$$s^2X(s) - s\big(x(0)\big) - x'(0) + X(s) = \frac{s}{s^2+9}$$

$$s^2X(s) - s + X(s) = \frac{s}{s^2+9}\ ;$$ Now solve for $X(s)$.

$$X(s)(s^2+1) = \frac{s}{s^2+9} + s = \frac{s^3+10s}{s^2+9}$$

$$X(s) = \frac{s^3+10s}{(s^2+9)(s^2+1)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+1}$$ Use partial Fractions.

$$\frac{s^3+10s}{(s^2+9)(s^2+1)} = \frac{(As+B)\left(s^2+1\right)+(Cs+D)(s^2+9)}{(s^2+9)(s^2+1)}$$

$$s^3 + 10s = (A + C)s^3 + (B + D)s^2 + (A + 9C)s + (B + 9D)$$

$$A + C = 1\,; \qquad A + 9C = 10$$

$$B + D = 0\,; \qquad B + 9D = 0$$

So we know $B = D = 0$

and $A + C = 1$, $A + 9C = 10$, so $A = -\frac{1}{8}$, $C = \frac{9}{8}$.

$$X(s) = -\frac{1}{8}\left(\frac{s}{s^2+9}\right) + \frac{9}{8}\left(\frac{s}{s^2+1}\right)$$

$$x(t) = \mathcal{L}^{-1}\left(X(s)\right) = -\frac{1}{8}\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) + \frac{9}{8}\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)$$

$$x(t) = -\frac{1}{8}\cos 3t + \frac{9}{8}\cos t.$$

Laplace transforms can also be used to solve simultaneous differential equations. That is a system of differential equations with more than one unknown function.