Laplace Transforms

We have seen how to solve differential equations of the forms:

$$mx'' + cx' + kx = F(t) \quad \text{and} \quad LI'' + RI' + \frac{1}{C}I = E'(t)$$

corresponding to a mass-spring-dashpot system and a series RLC circuit. However, it often happens in practice that the forcing term, $F(t)$ or $E'(t)$, is discontinuous. In that case, the earlier methods can be very messy. In these cases, Laplace transform methods can be easier.

We are already familiar with some functions (called operators) that map functions into functions (for example, taking a derivative does this). If we start with the function $f(x) = x^3$ and apply the derivative, we get $f'(x) = 3x^2$.

$$D\colon\ f(x) \to f'(x).$$

Similarly, a Laplace transform will map a function to another function by the formula:

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

for all values of s for which the improper integral converges.

Ex. Find the Laplace transform of the function $f(t) = 1$ for $t \geq 0$.

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt = \int_0^\infty e^{-st} dt = \lim_{b\to\infty} \int_0^b e^{-st} dt$$

$$= \lim_{b\to\infty} -\frac{1}{s} e^{-st}\Big|_{t=0}^{t=b} = \lim_{b\to\infty} \left(-\frac{1}{s} e^{-bs} + \frac{1}{s} e^0\right)$$

$$= \frac{1}{s} \qquad \text{for } s > 0.$$

(Notice for $s < 0$, $\lim_{b\to\infty} -\frac{1}{s} e^{-bs} = +\infty$, and if $s = 0$, $\int_0^\infty dt = +\infty$.)

Ex. Find the Laplace transform of $f(t) = e^{at}$, where a is a constant.

$$\mathcal{L}(e^{at}) = \int_0^{\infty} e^{-st} e^{at}\, dt = \int_0^{\infty} e^{-(s-a)t}\, dt = -\frac{1}{s-a}\left(e^{-(s-a)t}\right)\Big|_{t=0}^{t=\infty}$$

If $s - a > 0$ then $\lim_{t\to\infty} e^{-(s-a)t} = 0$;

if $s - a < 0$ then $\lim_{t\to\infty} e^{-(s-a)t} = \infty$;

if $s = a$ then $\int_0^{\infty} e^{-(s-a)t}\, dt = \int_0^{\infty} 1\, dt = \infty$.

So for $s > a$:

$$\mathcal{L}(e^{at}) = \lim_{b\to\infty}\left[-\frac{1}{s-a}\left(e^{-(s-a)b} - e^{0}\right)\right]$$

$$F(s) = \mathcal{L}(e^{at}) = \frac{1}{s-a} \quad \text{for } s > a.$$

Ex. Find the Laplace transform of $f(t) = t^a$, a is real number, $a > -1$.

$$\mathcal{L}(t^a) = \int_{t=0}^{t=\infty} e^{-st} t^a\, dt = \int_{u=0}^{u=\infty} e^{-u}\left(\frac{u}{s}\right)^a \left(\frac{1}{s}\right) du\,; \quad \text{if } s > 0$$

Let $u = st$, so $t = \frac{u}{s}$

$du = s\, dt$ so $\frac{1}{s}\, du = dt$

$$\mathcal{L}(t^a) = \frac{1}{s^{a+1}} \int_{u=0}^{u=\infty} e^{-u} u^a\, du = \frac{\Gamma(a+1)}{s^{a+1}}; \quad s > 0.$$

Recall that if n is a positive integer then $\Gamma(n) = (n-1)!$. Thus we have:

$$\mathcal{L}(t) = \frac{\Gamma(2)}{s^2} = \frac{1!}{s^2} = \frac{1}{s^2}$$

$$\mathcal{L}(t^2) = \frac{\Gamma(3)}{s^3} = \frac{2!}{s^3} = \frac{2}{s^3}$$

$$\mathcal{L}(t^3) = \frac{\Gamma(4)}{s^4} = \frac{3!}{s^4} = \frac{6}{s^4}$$

$$\mathcal{L}(t^n) = \frac{\Gamma(\mathrm{n+1})}{s^{n+1}} = \frac{n!}{s^{n+1}}; \quad n \in \mathbb{Z}^+.$$

You can also get these formulas without the gamma function by integrating by parts.

Theorem: Linearity of the Laplace Transform

If a and b are constants, then:

$$\mathcal{L}\big(af(t) + bg(t)\big) = a\mathcal{L}\big(f(t)\big) + b\mathcal{L}(g(t))$$

For all s, such that the Laplace transform of the functions f and g both exist.

Proof: $\mathcal{L}\big(af(t) + bg(t)\big) = \int_0^\infty e^{-st}(a\, f(t) + b\, g(t))\ dt$

$$= a\int_0^\infty e^{-st}\, f(t)\, dt + b\int_0^\infty e^{-st}\, g(t)\, dt$$

$$= a\,\mathcal{L}\big(f(t)\big) + b\,\mathcal{L}\big(g(t)\big).$$

Ex. Calculate $\mathcal{L}\left(2t^4 - 3t^{\frac{5}{2}}\right)$.

$$\mathcal{L}\left(2t^4 - 3t^{\frac{5}{2}}\right) = 2\mathcal{L}(t^4) - 3\mathcal{L}\left(t^{\frac{5}{2}}\right)$$

$$\mathcal{L}(t^a) = \frac{\Gamma(a+1)}{s^{a+1}}; \quad s > 0$$

$$\mathcal{L}\left(2t^4 - 3t^{\frac{5}{2}}\right) = 2\left(\frac{\Gamma(5)}{s^5}\right) - 3\left(\frac{\Gamma\left(\frac{7}{2}\right)}{s^{\frac{7}{2}}}\right)$$

$$\Gamma(5) = 4! = 24.$$

To calculate $\Gamma(\frac{7}{2})$ we need to use:

$$\Gamma(x+1) = x\Gamma(x) \text{ and } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2} + 1\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right)$$

$$= \frac{5}{2}\Gamma\left(\frac{3}{2} + 1\right) = \frac{5}{2}\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)$$

$$= \frac{15}{4}\Gamma(\frac{1}{2} + 1)$$

$$= \frac{15}{4}\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{15}{8}\sqrt{\pi}.$$

So $\mathcal{L}\left(2t^4 - 3t^{\frac{5}{2}}\right) = 2\left(\frac{24}{s^5}\right) - 3\left(\frac{\frac{15}{8}\sqrt{\pi}}{s^{\frac{7}{2}}}\right)$

$$= \frac{48}{s^5} - \frac{45}{8}\sqrt{\pi}\left(\frac{1}{s^{\frac{7}{2}}}\right); \quad s > 0.$$

Ex. Calculate $\mathcal{L}(\cosh kt)$ and $\mathcal{L}(\sinh kt)$, $k > 0$.

Recall: $\cosh kt = \frac{e^{kt}+e^{-kt}}{2}$, $\sinh kt = \frac{e^{kt}-e^{-kt}}{2}$, $\mathcal{L}(e^{at}) = \frac{1}{s-a}$, $s > a$.

$$\mathcal{L}(\cosh kt) = \mathcal{L}\left(\frac{e^{kt}+e^{-kt}}{2}\right) = \frac{1}{2}\left[\mathcal{L}(e^{kt}) + \mathcal{L}(e^{-kt})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-k} + \frac{1}{s+k}\right]$$

$$= \frac{s}{s^2-k^2}; \quad s > k > 0.$$

Similarly:

$$\mathcal{L}(\sinh kt) = \mathcal{L}\left(\frac{e^{kt}-e^{-kt}}{2}\right) = \frac{1}{2}\left[\frac{1}{s-k} - \frac{1}{s+k}\right]$$

$$= \frac{k}{s^2-k^2}; \quad s > k > 0.$$

Notice that since $e^{ikt} = \cos kt + i\sin kt$ and $e^{-ikt} = \cos kt - i\sin kt$:

$$\cos kt = \frac{e^{ikt}+e^{-ikt}}{2} \quad \text{and} \quad \sin kt = \frac{e^{ikt}-e^{-ikt}}{2i}.$$

By direct calculation it can be shown that $\mathcal{L}(e^{ikt}) = \frac{1}{s-ik}$, $s > 0$.

$$\mathcal{L}(\cos kt) = \frac{1}{2}\left[\mathcal{L}(e^{ikt}) + \mathcal{L}(e^{-ikt})\right] = \frac{1}{2}\left(\frac{1}{s-ik} + \frac{1}{s+ik}\right)$$

$$= \frac{1}{2}\left(\frac{(s+ik)+(s-ik)}{(s-ik)(s+ik)}\right) = \frac{1}{2}\left(\frac{2s}{s^2+k^2}\right) = \frac{s}{s^2+k^2}; \quad s > 0.$$

Similarly:

$$\mathcal{L}(\sin kt) = \frac{1}{2i}\left[\mathcal{L}(e^{ikt}) - \mathcal{L}(e^{-ikt})\right] = \frac{1}{2i}\left(\frac{1}{s-ik} - \frac{1}{s+ik}\right)$$

$$= \frac{k}{s^2+k^2}; \quad s > 0.$$

$\mathcal{L}(\cos kt)$ and $\mathcal{L}(\sin kt)$ can also be calculated directly from the definition of a Laplace transform by integrating by parts twice.

Ex. Find the Laplace transform of $f(t) = 2e^{-3t} - 4(\sin 2t)\,(\cos 2t)$.

$$\begin{aligned}\mathcal{L}\big(f(t)\big) &= 2\mathcal{L}(e^{-3t}) - 2\mathcal{L}[(2\sin 2t)(\cos 2t)] \\ &= 2\mathcal{L}(e^{-3t}) - 2\mathcal{L}(\sin 4t) \\ &= 2\left(\frac{1}{s+3}\right) - 2\left(\frac{4}{s^2+16}\right)\ ;\ \ s > 0.\end{aligned}$$

$$\mathcal{L}(2e^{-3t} - 4(\sin 2t)\,(\cos 2t)) = \frac{2}{s+3} - \frac{8}{s^2+16};\quad s > 0.$$

No two different continuous functions for $t \geq 0$ have the same Laplace transform. Thus, if $F(s)$ is the transform of a continuous function, $f(t)$, then $f(t)$ is uniquely determined. If $F(s) = \mathcal{L}(f(t))$, then $f(t) = \mathcal{L}^{-1}(F(s))$; where $\mathcal{L}^{-1}$ is the inverse Laplace transform.

Ex. Find $\mathcal{L}^{-1}\left(\frac{1}{s^4}\right)$, $\mathcal{L}^{-1}\left(\frac{1}{s-4}\right)$, $\mathcal{L}^{-1}(\frac{4s}{s^2+4})$ using the fact that:

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}\ ;\ \ n \geq 0\ ;\quad \mathcal{L}(e^{at}) = \frac{1}{s-a};\qquad \mathcal{L}(\cos kt) = \frac{s}{s^2+k^2}\,.$$

$$\mathcal{L}(t^3) = \frac{6}{s^4}\ ,\qquad\qquad \mathcal{L}(\cos 2t) = \frac{s}{s^2+2^2} = \frac{s}{s^2+4}$$

$$\mathcal{L}\left(\frac{1}{6}t^3\right) = \frac{1}{s^4},\qquad \mathcal{L}(e^{4t}) = \frac{1}{s-4},\qquad \mathcal{L}(4\cos 2t) = \frac{4s}{s^2+4}$$

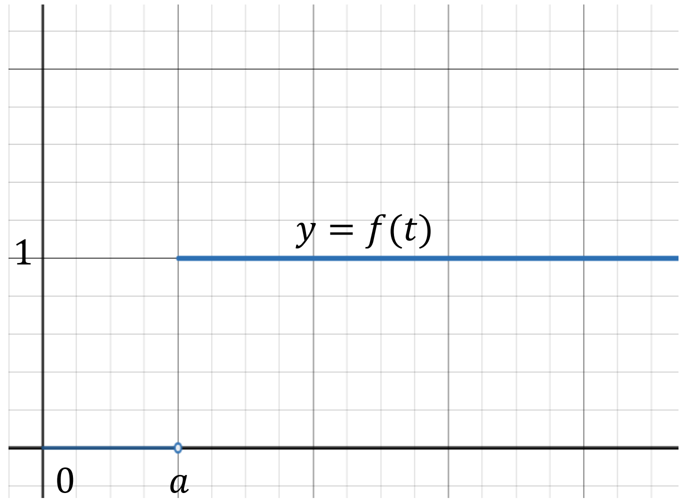
$$\mathcal{L}^{-1}\left(\frac{1}{s^4}\right) = \frac{1}{6}t^3,\quad \mathcal{L}^{-1}\left(\frac{1}{s-4}\right) = e^{4t}\ ,\qquad \mathcal{L}^{-1}\left(\frac{4s}{s^2+4}\right) = 4\cos 2t.$$

Piecewise Continuous Functions

Ex. Let $f(t) = 0 \quad$ if $t < a$

$\qquad = 1 \quad$ if $t \geq a$

where $a \geq 0$. Find $\mathcal{L}(f(t))$.



$$\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t)dt = \int_a^{\infty} e^{-st}dt$$

$$= -\frac{1}{s}e^{-st}\Big|_{t=a}^{t=\infty}$$

$$= \lim_{b\to\infty} -\frac{1}{s}e^{-st}\Big|_{t=a}^{t=b}$$

$$= \lim_{b\to\infty} -\frac{1}{s}(e^{-sb} - e^{-sa})$$

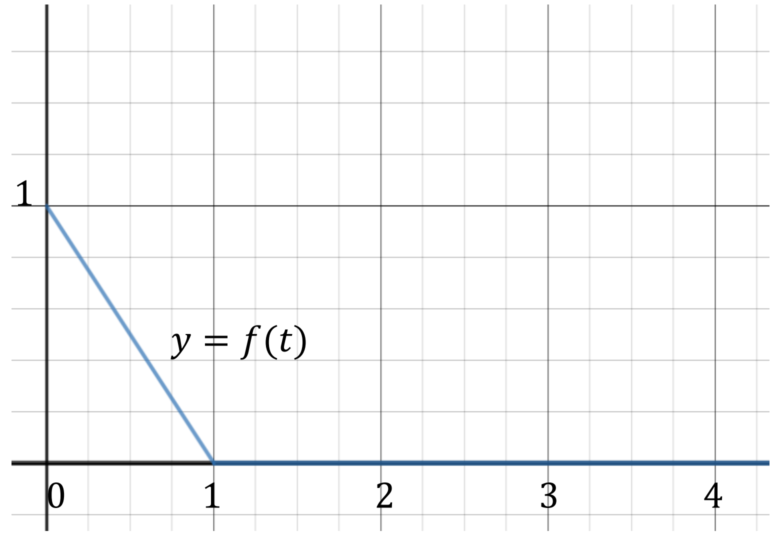
$$= \frac{e^{-as}}{s}\ ; \qquad s > 0.$$

$f(t)$, in this case, is also written as $u(t-a)$, the unit step function. So we can write:

$$\mathcal{L}(u(t-a)) = \frac{e^{-as}}{s}\ ; \qquad s > 0.$$

Ex. Find $\mathcal{L}(f(t))$ when

$$f(t) = 1 - t \quad \text{if } 0 \le t \le 1$$
$$= 0 \quad \text{if } t > 1$$



$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt$$
$$= \int_0^1 e^{-st}(1-t)dt$$
$$= \int_0^1 e^{-st}dt - \int_0^1 t\, e^{-st}dt$$
$$= -\frac{1}{s}e^{-st}\Big|_{t=0}^{t=1} - \int_0^1 t\, e^{-st}dt$$

Now integrate by parts.

Let $u = t \qquad v = -\frac{1}{s}e^{-st}$

$du = dt \qquad dv = e^{-st}$

$$= -\frac{1}{s}(e^{-s}-1) - [-\frac{1}{s}e^{-st}(t)\Big|_{t=0}^{t=1} - \int_0^1 -\frac{1}{s}e^{-st}dt]$$
$$= -\frac{1}{s}(e^{-s}-1) - [-\frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-st}\Big|_{t=0}^{t=1}]$$
$$= \frac{1}{s} + \frac{1}{s^2}(e^{-s} - e^0)$$
$$= \frac{1}{s} + \frac{e^{-s}}{s^2} - \frac{1}{s^2}$$
$$\mathcal{L}(f(t)) = \frac{s-1+e^{-s}}{s^2}.$$