The Gamma Function and Bessel Functions

Bessel's equation of order $p \geq 0$ is:

$$x^2y'' + xy' + (x^2 - p^2)y = 0.$$

Solutions are called Bessel functions of order p.

$$y'' + \frac{1}{x}y' + \frac{x^2-p^2}{x^2}y = 0.$$

So $p(x) = 1, \; p(0) = 1$; $\quad q(x) = x^2 - p^2, \; q(0) = -p^2$.

Indicial equation:

$$r(r-1) + r - p^2 = 0$$

$$r^2 - p^2 = 0$$

$$r = \pm p.$$

If we substitute $y = \sum_{m=0}^{\infty} c_m x^{m+r}$ into $x^2y'' + xy' + (x^2 - p^2)y = 0$:

$$x^2 \sum_{m=0}^{\infty}(m+r)(m+r-1)c_m x^{m+r-2} + x\sum_{m=0}^{\infty}(m+r)c_m x^{m+r-1} + (x^2-p^2)\sum_{m=0}^{\infty} c_m x^{m+r} = 0$$

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)c_m x^{m+r} + \sum_{m=0}^{\infty}(m+r)c_m x^{m+r} + \sum_{m=0}^{\infty} c_m x^{m+r+2} - \sum_{m=0}^{\infty} p^2 c_m x^{m+r} = 0$$

$$\sum_{m=0}^{\infty}[(m+r)(m+r-1) + (m+r) - p^2]c_m x^{m+r} + \sum_{m=2}^{\infty} c_{m-2} x^{m+r} = 0$$

$$\sum_{m=0}^{\infty}[(m+r)^2 - p^2]c_m x^{m+r} + \sum_{m=2}^{\infty} c_{m-2} x^{m+r} = 0.$$

$m = 0$: $(r^2 - p^2)c_0 = 0$

so c_0 can be any number because $r^2 - p^2 = 0$.

$m = 1$: $[(1 + r)^2 - p^2]c_1 = 0$

since if $r = \pm p$ then $(1 + r)^2 - p^2 \neq 0$, $\Rightarrow c_1 = 0$, unless $r = -\frac{1}{2}$.

$m \geq 2$: $[(m + r)^2 - p^2]c_m + c_{m-2} = 0$

$$c_m = -\frac{c_{m-2}}{(m+r)^2-p^2}.$$

Case where $r = p > 0$:

$$(m + p)^2 - p^2 = m^2 + 2mp$$

So using a_m for c_m, in this case:

$$a_m = -\frac{a_{m-2}}{m^2+2mp} = -\frac{a_{m-2}}{m(2p+m)}\ ; \quad \text{for } m \geq 2.$$

Because $a_1 = 0$, all odd a_ms are also 0.

$m = 2$: $a_2 = -\frac{a_0}{2(2p+2)}$

$m = 4$: $a_4 = -\frac{a_2}{4(2p+4)} = \frac{a_0}{2(4)(2p+2)(2p+4)}$

$m = 6$: $a_6 = -\frac{a_4}{6(2p+6)} = -\frac{a_0}{2(4)(6)(2p+2)(2p+4)(2p+6)}$

$$\Rightarrow \quad a_{2m} = \frac{(-1)^m a_0}{2(4)(6)\cdots(2m)(2p+2)(2p+4)\cdots(2p+2m)}$$

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m}(m)!(p+1)(p+2)\cdots(p+m)}$$

$$y_1(x) = a_0 \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+p}}{2^{2m}(m)!(p+1)\ldots(p+m)}.$$

We saw when $p = 0$, this is the only Frobenius series solution.

Case where $r = -p < 0$

$(m-p)^2 - p^2 = m^2 - 2mp = m(m-2p)$

Now using b_m for c_m we get:

$$b_m = -\frac{b_{m-2}}{m(m-2p)}; \quad \text{for } m \geq 2.$$

Once again, b_0 can be any constant and $b_1 = 0$, unless $r = -\frac{1}{2}$.

Notice if p is either a positive integer or an odd multiple ($\geq$ 3) of $\frac{1}{2}$, then $m - 2p$ will equal 0 for some positive integer m. If that happens there will be no solution for $m(m-2p)b_m + b_{m-2} = 0$ if $b_{m-2} \neq 0$.

If $p = \frac{k}{2}$, k an odd positive integer, then we don't have a problem because we can choose $b_m = 0$ for all odd m. So if p is not a positive integer, we have:

$$b_m = -\frac{b_{m-2}}{m(m-2p)} \; ; \quad m \geq 2$$

And we get:

$$y_2(x) = b_0 \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-p}}{2^{2m} m!(-p+1)(-p+2)\ldots(-p+m)}.$$

If $r = -\frac{1}{2}$ then b_1 is an arbitrary constant and

$$y_2(x) = b_0 \left(x^{-\frac{1}{2}}\right) cosx + b_1 \left(x^{-\frac{1}{2}}\right) sinx.$$

But if $r = \frac{1}{2}$, $y_1(x) = a_0 \left(x^{-\frac{1}{2}}\right) sinx$, thus we can take $b_1 = 0$ above.

The series representation of $y_1(x)$ and $y_2(x)$ converge for $x > 0$, since $x = 0$ is the only singular point of the differential equation.

The gamma function

Def. The gamma function is defined as:

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt, \quad x > 0.$$

a) $\Gamma(1) = \int_0^\infty e^{-t}\ dt = \lim_{b\to\infty} (-e^{-t}\,|_{t=0}^{t=b}) = 1.$

b) $\Gamma(x+1) = x\Gamma(x)$

Proof:

$$\Gamma(x+1) = \lim_{b\to\infty} \int_0^b e^{-t} t^x\ dt;$$ integrate by parts:

Let $u = t^x$ $\qquad v = -e^{-t}$

$du = xt^{x-1}$ $\qquad dv = e^{-t} dt$

$$= \lim_{b\to\infty} [-t^x e^{-t}|_{t=0}^{t=b} + x\int_0^b e^{-t} t^{x-1} dt] = x\Gamma(x)$$

Since $\lim_{b\to\infty} \frac{-b^x}{e^b} = 0$ for any $x > 0$.

So $\Gamma(x+1) = x\Gamma(x)$. In particular:

$\Gamma(2) = 1\Gamma(1) = 1!$

$\Gamma(3) = 2\Gamma(2) = 2!$

$\Gamma(4) = 3\Gamma(3) = 3!$

$\Longrightarrow \Gamma(n+1) = n!$ for $n \geq 0$ an integer.

An important special value of the gamma function is:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{-\frac{1}{2}}\, dt$$

Let $u^2 = t$

$2u\, du = dt$ or $2du = \frac{1}{u} dt = \frac{1}{\sqrt{t}} dt$

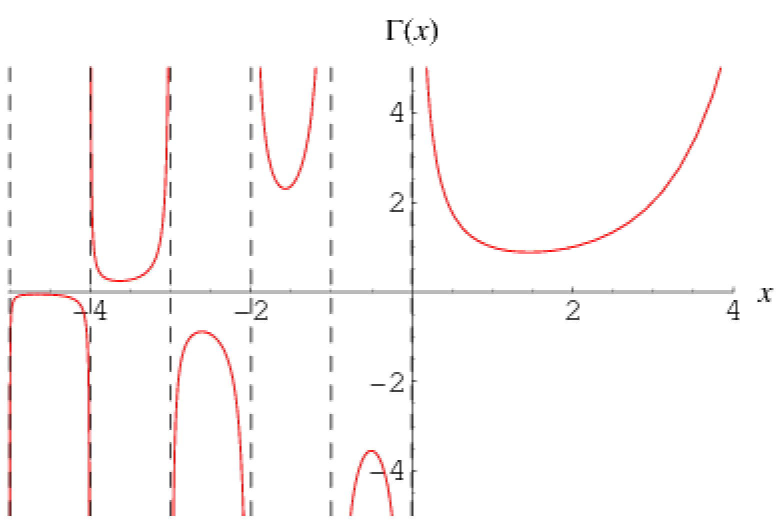
$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2}\, du = \sqrt{\pi}\,.$$

The gamma function, $\Gamma(x)$, is defined for $x > 0$. However, we can extend its definition to all $x < 0$ such that x is not a negative integer. We can do this through the relationship:

$$\Gamma(x+1) = x\Gamma(x) \quad \Rightarrow \quad \Gamma(x) = \frac{\Gamma(x+1)}{x}.$$

For example, if $0 < x + 1$, then $\Gamma(x+1)$ is defined. We can then define $\Gamma(x) = \frac{\Gamma(x+1)}{x}$ for $-1 < x < 0$. Now that $\Gamma(x)$ is defined for $-1 < x < 0$, we can define $\Gamma(x)$, for $-2 < x < -1$ by $\Gamma(x) = \frac{\Gamma(x+1)}{x}$. Continuing this process we get a definition of $\Gamma(x)$ for all $x < 0$ such that x is not a negative integer.

Below is a graph of the function $\Gamma(x)$.



Bessel functions of the first kind

The solution to $x^2y'' + xy' + (x^2 - p^2)y = 0$ corresponding to $r = p > 0$ is

$$y_1(x) = a_0 \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+p}}{2^{2m}(m!)(p+1)\dots(p+m)} .$$

If we choose $a_0 = \frac{1}{2^p\Gamma(p+1)}$; $p > 0$ and use:

$$\Gamma(p + m + 1) = (p + m)(p + m - 1) \dots (p + 2)(p + 1)\Gamma(\mathrm{p} + 1)$$

which follows from $\Gamma(x + 1) = x\Gamma(x)$ we get:

$$y_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+p}}{2^{2m}(2^p)(m!)(\Gamma(p+1))(p+1)\dots(p+m)}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+p}}{2^{2m+p}(m!)\Gamma(m+p+1)}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)\Gamma(m+p+1)} \left(\frac{x}{2}\right)^{2m+p} .$$

This is called **the Bessel function of the first kind of order p** denoted by:

$$J_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)\Gamma(m+p+1)} \left(\frac{x}{2}\right)^{2m+p} .$$

Similarly, if $p > 0$ is not an integer, we choose $b_0 = \frac{1}{2^{-p}\Gamma(-p+1)}$ in the Frobenius solution corresponding to $r = -p$ and get:

$$y_2(x) = b_0 \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-p}}{2^{2m}(m!)(-p+1)\ldots(-p+m)}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-p}}{2^{2m}(2^{-p})(m!)(\Gamma(-p+1))(-p+1)\ldots(-p+m)}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)\Gamma(-p+m+1)} \left(\frac{x}{2}\right)^{2m-p} = J_{-p}(x).$$

So if p is not an integer we have a general solution to Bessel's equation of order p:

$$y(x) = c_1 J_p(x) + c_2 J_{-p}(x); \qquad x > 0.$$

To get the correct solution for $x < 0$ we need to replace x^p with $|x|^p$ in $J_p(x)$ and $J_{-p}(x)$.

If $p = n$; a nonnegative integer, then

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2m+n}$$

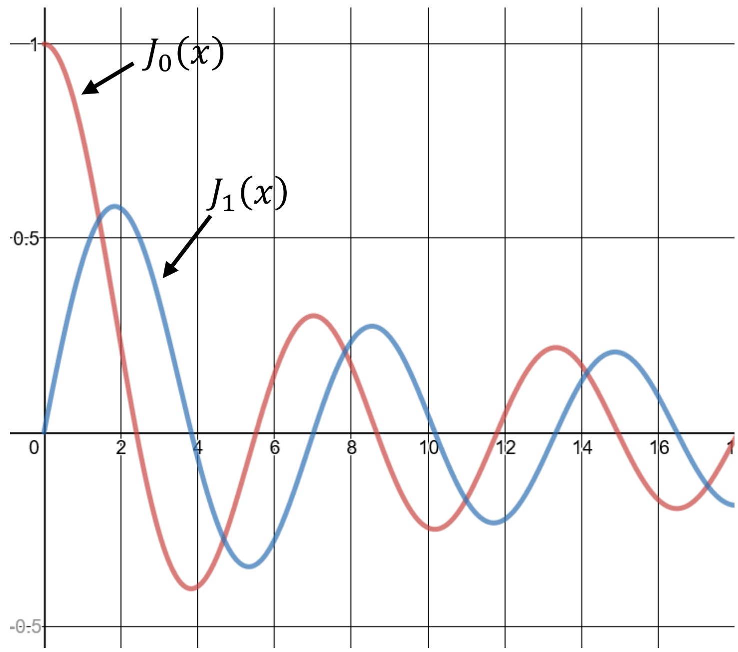
$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)(n+m)!} \left(\frac{x}{2}\right)^{2m+n}$$

So we have:

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2(4^2)} - \frac{x^6}{2^2(4^2)(6^2)} + \cdots$$

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1}(m!)(m+1)!} = \frac{x}{2} - \frac{1}{2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!(3!)}\left(\frac{x}{2}\right)^5 + \cdots.$$

The graph of $J_0(x)$ looks a bit like a damped graph of $\cos x$ and the graph of $J_1(x)$ looks a bit like a damped graph of $\sin x$.



Similar to the relationship between $\cos x$ and $\sin x$,

$$J_0'(x) = -J_1(x).$$

Bessel functions of the second kind

If p is an integer, n, then the solution to

$$x^2y'' + xy' + (x^2 - p^2)y = 0$$

has $y_1(x)$ as a solution, but there is no second Frobenius series. The general solution is:

$$y(x) = c_1J_n(x) + c_2Y_n(x)$$

where,

$$Y_n(x) = \frac{2}{\pi}(\gamma + \ln(\tfrac{x}{2}))\,J_n(x) - \frac{1}{\pi}\sum_{m=0}^{n-1}\frac{2^{n-2m}(n-m-1)!}{m!x^{n-2m}}$$

$$-\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m(H_m+H_{m+n})}{m!(m+n)!}\left(\frac{x}{2}\right)^{n+2m}$$

where $\gamma = \lim_{n\to\infty}(\sum_{k=1}^{n}\frac{1}{k} - \ln n)$ (Euler's constant) and

$H_m = \sum_{k=1}^{m}\frac{1}{k}$.

$Y_n(x)$ is called the Bessel function of the second kind of integral order.

Bessel function Relationships

$$J_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(p+m+1)} \left(\frac{x}{2}\right)^{2m+p}$$

If p is a nonnegative integer then:

$$\frac{d}{dx}[x^p J_p(x)] = \frac{d}{dx}(\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2p}}{2^{2m+p}(m!)(p+m)!})$$

$$= \sum_{m=0}^{\infty} \frac{(2m+2p)(-1)^m x^{2m+2p-1}}{2^{2m+p}(m!)(p+m)!}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2p-1}}{2^{2m+p-1}(m!)(p+m-1)!}$$

$$= x^p \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+p-1}}{2^{2m+p-1}(m!)(p+m-1)!})$$

$$= x^p J_{p-1}(x)$$

So $\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x).$

Similarly, $\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x).$

Applying the product rule to $\frac{d}{dx}[x^p J_p(x)]$ and $\frac{d}{dx}[x^{-p} J_p(x)]$ we get:

$$J_p'(x) = J_{p-1}(x) - \frac{p}{x} J_p(x)$$

$$J_p'(x) = \frac{p}{x} J_p(x) - J_{p+1}(x).$$

Subtracting these equations, we get:

$$0 = J_{p-1}(x) - \frac{2p}{x} J_p(x) + J_{p+1}(x)$$

a recurrence relationship between Bessel functions of the first kind.

Ex. With $p = 1$ in $\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$, we get:

$$\frac{d}{dx}[xJ_1(x)] = xJ_0(x)$$

$$\int \frac{d}{dx}[xJ_1(x)] = \int xJ_0(x)\, dx$$

$$xJ_1(x) + C = \int xJ_0(x)\, dx.$$

Ex. With $p = 0$ in $\frac{d}{dx}\left[x^{-p}J_p(x)\right] = -x^{-p}J_{p+1}(x)$, we get:

$$\frac{d}{dx}[J_0(x)] = -J_1(x)$$

$$\int \frac{d}{dx}[J_0(x)] = \int -J_1(x)\, dx$$

$$J_0(x) + C = -\int J_1(x)\, dx$$

$$-J_0(x) + C = \int J_1(x)\, dx.$$

Ex. Find $\int x^2 J_0(x)\, dx$ and $\int x^3 J_0(x)\, dx$ in terms of Bessel functions and $\int J_0(x)\, dx$.

Integrate by parts and use the previous examples:

$$\int x^2 J_0(x)\, dx = x^2 J_1(x) - \int x J_1(x)\, dx$$

Let $u = x$ $\quad v = x\, J_1(x)$ $\qquad$ Let $u = x$ $\quad v = -J_0(x)$

$du = dx$ $\quad dv = xJ_0(x)dx$ $\qquad$ $du = dx$ $\quad dv = J_1(x)dx$

$$= x^2 J_1(x) - \left[-xJ_0(x) - \int -J_0(x)\, dx\right]$$

$$\int x^2 J_0(x)\, dx = x^2 J_1(x) + xJ_0(x) - \int J_0(x)\, dx.$$

$$\int x^3 J_0(x)\,dx = x^3 J_1 - \int 2x^2 J_1(x)\,dx$$

Let $u = x^2$ $\quad v = x\,J_1$ $\qquad$ Let $u = 2x^2$ $\quad v = -J_0(x)$

$du = 2x\,dx$ $\quad dv = xJ_0 dx$ $\qquad du = 4x\,dx$ $\quad dv = J_1(x)dx$

$$= x^3 J_1 - [-2x^2 J_0(x) - 4\int -xJ_0(x)\,dx]$$

$$\int x^3 J_0(x)\,dx = x^3 J_1(x) + 2x^2 J_0(x) - 4xJ_1(x) + C.$$