Series Solutions Near Regular Singular Points: $r_1 - r_2$ is an integer

Consider the differential equation:

$$y'' + \frac{p(x)}{x}y' + \frac{q(x)}{x^2}y = 0$$

where $p(x)$ and $q(x)$ are analytic at $x = 0$ and thus $x = 0$ is a regular singular point.

We had a theorem that said there are two linearly independent solutions through a Frobenius series if $r_1 \neq r_2$ and $r_1 - r_2$ is not a positive integer ($r_1 \geq r_2$ are the real roots of the indicial equation: $r(r-1) + p_0 r + q_0 = 0$).

When $r_1 = r_2$ there can only be one Frobenius series solution.

When $r_1 - r_2 = N$, a positive integer, then there may (or may not) be two linearly independent Frobenius series solutions.

The Nonlogarithmic Case with $r_1 = r_2 + N$

Ex. Solve $xy'' + (3 - x)y' - y = 0$.

Dividing by x we get:

$$y'' + \frac{3-x}{x}y' - \frac{x}{x^2}y = 0$$

So $p(x) = 3 - x$ and $p(0) = 3$

$q(x) = -x$ and $q(0) = 0$.

The indicial equation becomes:

$$r(r-1)+3r=0$$

$$r^2+2r=0$$

$$r(r+2)=0$$

$$r=0,-2.$$

So $r_1=0,\ r_2=-2$ and $r_1-r_2=2$, a positive integer.

Substituting $y=\sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation:

$$xy''+(3-x)y'-y=0$$

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} \\ +(3-x)\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}-\sum_{n=0}^{\infty} c_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1}+3\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} \\ -x\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}-\sum_{n=0}^{\infty} c_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1}+\sum_{n=0}^{\infty} 3(n+r)c_n x^{n+r-1} \\ -\sum_{n=0}^{\infty}(n+r)c_n x^{n+r}-\sum_{n=0}^{\infty} c_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}[(n+r)(n+r-1)+3(n+r)]c_n x^{n+r-1} \\ -\sum_{n=0}^{\infty}(n+r+1)c_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}[(n+r)^2+2(n+r)]c_n x^{n+r-1}-\sum_{n=1}^{\infty}(n+r)c_{n-1}x^{n+r-1}=0.$$

The term corresponding to $n = 0$ is the indicial equation:

$n = 0 :\quad (r^2 + 2r)c_0 = 0$ or $r(r+2)c_0 = 0$.

Since $r(r+2) = 0$ for both roots r_1 and r_2, c_0 is an arbitrary constant.

For $n \geq 1 :\quad [(n+r)^2 + 2(n+r)]c_n - (n+r)c_{n-1} = 0$

$$(n+r)(n+r+2)c_n - (n+r)c_{n-1} = 0$$

In this case, start with the smaller root, $r_2 = -2$.

$$(n-2)(n)c_n - (n-2)c_{n-1} = 0.$$

If $n \neq 2$, we can solve for c_n:

$$c_n = \frac{c_{n-1}}{n}.$$

So we get:

$$n = 1 \qquad\qquad c_1 = c_0$$

When we have $r_1 = r_2 + N$, it will always be the coefficient c_N that requires special consideration. In this example $N = 2$.

If $n = 2$, then $(n-2)(n)c_n - (n-2)c_{n-1} = 0; \qquad 0c_2 - 0c_1 = 0.$

We know that $c_1 = c_0 \neq 0$, but c_2 can be anything.

So c_2 is a second arbitrary constant along with c_0.

Continuing to use the recursion formula: $c_n = \frac{c_{n-1}}{n}$; $n > 2$:

$$n = 3 \qquad c_3 = \frac{c_2}{3}$$

$$n = 4 \qquad c_4 = \frac{c_3}{4} = \frac{c_2}{3(4)}$$

$$n = 5 \qquad c_5 = \frac{c_4}{5} = \frac{c_2}{3(4)(5)} = \frac{2c_2}{5!}.$$

$$c_n = \frac{2c_2}{n!}; \qquad n > 2.$$

So $y = x^r \sum_{n=0}^{\infty} c_n x^n = x^{-2} \sum_{n=0}^{\infty} c_n x^n$

$$y = c_0 x^{-2}(1 + x) + c_2 x^{-2}\left(x^2 + \sum_{n=3}^{\infty} \frac{2x^n}{n!}\right)$$

$$= c_0 \frac{(1+x)}{x^2} + c_2\left(1 + \sum_{n=1}^{\infty} \frac{2x^n}{(n+2)!}\right).$$

So we have two linearly independent solutions:

$$y_1(x) = x^{-2}(1 + x)$$

$$y_2(x) = 1 + \frac{2}{3!}x + \frac{2}{4!}x^2 + \frac{2}{5!}x^3 + \cdots + \frac{2}{(n+2)!}x^n + \cdots$$

$$y_2(x) = 1 + \sum_{n=1}^{\infty} \frac{2x^n}{(n+2)!}$$

Notice that we haven't found the Frobenius solution corresponding to the larger root $r_1 = 0$. However, if we did solve for the coefficients corresponding to $r_1 = 0$ we would find the series $y_2(x)$ above.

Now let's see an example where $r_1 - r_2 =$positive integer, but we don't get two linearly independent Frobenius series as solutions.

Ex. Consider $x^2y'' - xy' + (x^2 - 3)y = 0.$

Dividing by x^2 we get:

$$y'' - \frac{1}{x}y' + \frac{x^2-3}{x^2}y = 0$$

So $p(x) = -1$ and $p(0) = -1$

$q(x) = x^2 - 3$ and $q(0) = -3.$

The indicial equation becomes:

$$r(r-1) - r - 3 = 0$$

$$r^2 - 2r - 3 = 0$$

$$(r+1)(r-3) = 0$$

$$r = -1, 3$$

So $r_1 - r_2 = 3 - (-1) = 4$, a positive integer.

Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation:

$$x^2y'' - xy' + (x^2 - 3)y = 0$$

$$x^2 \sum_{n=0}^{\infty}(n+r-1)(n+r)c_n x^{n+r-2} - x\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + (x^2-3)\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r-1)(n+r)c_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} - \sum_{n=0}^{\infty} 3c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}[(n+r-1)(n+r)-(n+r)-3]c_n x^{n+r}+\sum_{n=0}^{\infty}c_n x^{n+r+2}=0$$

$$\sum_{n=0}^{\infty}[(n+r-1)(n+r)-(n+r)-3]c_n x^{n+r}+\sum_{n=2}^{\infty}c_{n-2} x^{n+r}=0$$

$$\sum_{n=0}^{\infty}[(n+r)^2-2(n+r)-3]c_n x^{n+r}+\sum_{n=2}^{\infty}c_{n-2} x^{n+r}=0.$$

When $n=0$ we get the indicial equation: $[(r-1)(r)-r-3]c_0=0.$

So c_0 can be any number.

If $n=1:$ $[(1+r)^2-2(1+r)-3]c_1=0$

Since $(1+r)^2-2(1+r)-3\neq 0$ for $r=-1,3$; $\Longrightarrow$ $c_1=0.$

For $n\geq 2:$ $[(n+r)^2-2(n+r)-3]c_n+c_{n-2}=0.$

Again, start with the smaller root $r=-1$

$$[(n+r)^2-2(n+r)-3]c_n+c_{n-2}=0$$

$$n(n-4)c_n+c_{n-2}=0\,;$$

$$\Longrightarrow \qquad c_n=-\frac{1}{n(n-4)}c_{n-2}\,;\;\; n\geq 2,\; n\neq 4.$$

Since $c_1=0$, all odd c_ns are 0.

$$n=2 \qquad\qquad c_2=\frac{c_0}{4}$$

If we have $n=4$, then we have: $4(4-4)c_4+c_2=0$. But $c_2=\frac{c_0}{4}\neq 0$ since $c_0\neq 0$. So there is no way to choose c_4 to satisfy this equation, thus there is no Frobenius series solution corresponding to the smaller root $r_2=-1$.

Now let's find the Frobenius series corresponding to the root $r_1 = 3$.

We substitute $r = 3$ into:

$$[(n+r)^2 - 2(n+r) - 3]c_n + c_{n-2} = 0$$

$$[(n+3)^2 - 2(n+3) - 3]c_n + c_{n-2} = 0$$

$$(n^2 + 4n)c_n + c_{n-2} = 0\,;$$

$$\Longrightarrow \qquad c_n = -\frac{1}{n(n+4)}c_{n-2}\,; \;\; n \geq 2: \quad \text{(odd } c_n\text{s are still 0)}$$

$$n = 2 \qquad c_2 = -\frac{c_0}{2(6)}$$

$$n = 4 \qquad c_4 = -\frac{c_2}{4(8)} = \frac{c_0}{2(4)(6)(8)}$$

$$n = 6 \qquad c_6 = -\frac{c_4}{6(10)} = -\frac{c_0}{2(4)(6)(6)(8)(10)}$$

$$\Longrightarrow \qquad c_{2n} = \frac{(-1)^n c_0}{2(4)(6)\cdots(2n)(6)(8)(10)\cdots(2n+4)}\,.$$

$$2(4)(6)\cdots(2n) = 2^n(1(2)(3)\cdots(n)) = 2^n(n!)$$

$$(6)(8)(10)\cdots(2n+4) = 2^n(3(4)(5)\cdots(n+2)) = \frac{2^n(n+2)!}{2}$$

$$\big(2(4)(6)\cdots(2n)\big)((6)(8)(10)\cdots(2n+4)) = 2^{2n-1}(n!)(n+2)!$$

$$\Longrightarrow \qquad c_{2n} = \frac{(-1)^n}{2^{2n-1}(n!)(n+2)!}c_0$$

So the Frobenius series solution is:

$$y_1(x) = c_0 x^3\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n-1}(n!)(n+2)!}\right).$$

When only one Frobenius solution exists, we need a way to find a second linearly independent solution. We do this through the method of reduction of order. We will use the fact that we know one solution, $y_1(x)$, to reduce a second order differential equation into a first order differential equation.

Suppose we know $y_1(x)$ is a solution to $y'' + P(x)y' + Q(x)y = 0.$

Let's say $y_2(x) = v(x)y_1(x)$. If we can find $v(x)$ then we know $y_2(x)$.

If $y_2(x)$ is also a solution to $y'' + P(x)y' + Q(x)y = 0$. Then,

$$y_2(x) = v(x)y_1(x)$$
$$y_2'(x) = v(x)y_1'(x) + v'(x)y_1(x)$$
$$y_2''(x) = v(x)y_1''(x) + v'(x)y_1'(x) + v'(x)y_1'(x) + v''(x)y_1(x)$$
$$y_2''(x) = v(x)y_1''(x) + 2v'(x)y_1'(x) + v''(x)y_1(x).$$

Substituting into $y'' + P(x)y' + Q(x)y = 0$:

$$(vy_1'' + 2v'y_1' + v''y_1) + P(x)(vy_1' + v'y_1) + Q(x)vy_1 = 0$$

Regrouping terms, we get:

$$v(y_1'' + P(x)y_1' + Q(x)y_1) + v''y_1 + 2v'y_1' + P(x)v'y_1 = 0.$$

But y_1 is a solution so $y_1'' + P(x)y_1' + Q(x)y_1 = 0$, so we can write

$$v''y_1 + 2v'y_1' + P(x)v'y_1 = 0$$
$$v''y_1 + (2y_1' + P(x)y_1)v' = 0.$$

Now let $u = v'$ so the equation becomes:

$$u'y_1 + (2y_1' + P(x)y_1)u = 0$$

$$u' + \left(\frac{2y_1'}{y_1} + P(x)\right)u = 0.$$

Thus an integrating factor for this equation is:

$$\rho = e^{\int\left(\frac{2y_1'}{y_1}+P(x)\right)dx} = e^{(2\ln|y_1| + \int P(x)dx)} = y_1^2 e^{\int P(x)dx}.$$

So, $y_1^2 e^{\int P(x)dx} u' + \left(2y_1'y_1 e^{\int P(x)dx} + y_1^2 P(x) e^{\int P(x)dx}\right)u = 0$

$$\left(uy_1^2 e^{\int P(x)dx}\right)' = 0$$

$$uy_1^2 e^{\int P(x)dx} = C$$

$$u = \frac{C}{y_1^2} e^{-\int P(x)dx}.$$

$$u = v' \quad \Longrightarrow \quad v' = \frac{C}{y_1^2} e^{-\int P(x)dx}$$

$$v = C\int \frac{e^{-\int P(x)dx}}{y_1^2}\, dx$$

$$v = \frac{y_2}{y_1} \quad \Longrightarrow \quad \frac{y_2}{y_1} = C\int \frac{e^{-\int P(x)dx}}{y_1^2}\, dx$$

$$y_2 = Cy_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx.$$

This reduction of order approach is used to find the second non-Frobenius solution described in the next theorem (The Logarithmic Case).

The Logarithmic Case

We now investigate the form of the second solution to:

$$y'' + \frac{p(x)}{x} y' + \frac{q(x)}{x^2} y = 0$$

under the assumption $r_1 = r_2 + N,\ N$ is a positive integer. We assume we have already found the Frobenius series solution:

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n\ ;\ \ a_0 \neq 0 \text{ for } x > 0.$$

Theorem: Suppose $x = 0$ is a regular singular point of

$$x^2 y'' + x p(x) y' + q(x) y = 0.$$

Let $\rho > 0$ be the minimum of the radii of convergence of

$$p(x) = \sum_{n=0}^{\infty} p_n x^n \text{ and } q(x) = \sum_{n=0}^{\infty} q_n x^n.$$

Let r_1, r_2 be the real roots of the indicial equation

$$r(r-1) + p_0 r + q_0 = 0\ ;\ \ r_1 \geq r_2$$

a) If $r_1 = r_2$ then the two solutions y_1 and y_2 are of the form

$$\begin{aligned} y_1 &= x^{r_1} \sum_{n=0}^{\infty} a_n x^n\ ; \quad a_0 \neq 0 \\ y_2 &= \big(y_1(x)\big) \ln x + x^{(r_1+1)} \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

b) If $r_1 - r_2 = N$, a positive integer, then

$$\begin{aligned} y_1 &= x^{r_1} \sum_{n=0}^{\infty} a_n x^n\ ; \quad a_0 \neq 0 \\ y_2 &= C\big(y_1(x)\big) \ln x + x^{r_2} \sum_{n=0}^{\infty} b_n x^n\ ;\ \ b_0 \neq 0. \end{aligned}$$

Note: C could be 0, in which case there are two Frobenius solutions. The radii of convergence of the power series in this theorem are at least ρ. The coefficients can be determined by direct substitution.

Ex. Derive the second solution of Bessel's equation of order zero

$$x^2y'' + xy' + x^2y = 0.$$

Recall $r_1 = r_2 = 0$ for this equation and so we can write:

$$y_1(x) = J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

Since $r_1 = r_2$,

$$y_2 = y_1(\ln x) + x^{(r_1+1)} \sum_{n=0}^{\infty} b_n x^n$$

$$y_2 = y_1(\ln x) + \sum_{n=0}^{\infty} b_n x^{n+1} = y_1 \ln x + \sum_{n=1}^{\infty} c_n x^n ; \qquad \text{since } r_1 = 0.$$

$$y_2' = y_1\left(\frac{1}{x}\right) + y_1'(\ln x) + \sum_{n=1}^{\infty} nc_n x^{n-1}$$

$$y_2'' = y_1\left(-\frac{1}{x^2}\right) + y_1'\left(\frac{1}{x}\right) + y_1''(\ln x) + y_1'\left(\frac{1}{x}\right) + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

$$= y_1''(\ln x) + \frac{2}{x}y_1' - \frac{y_1}{x^2} + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}.$$

Substituting into $x^2y'' + xy' + x^2y = 0$:

$$x^2\left[y_1\left(-\frac{1}{x^2}\right) + y_1'\left(\frac{2}{x}\right) + y_1''(\ln x) + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}\right] +$$
$$x\left[y_1\left(\frac{1}{x}\right) + y_1'(\ln x) + \sum_{n=1}^{\infty} nc_n x^{n-1}\right] + x^2\left(y_1 \ln x + \sum_{n=1}^{\infty} c_n x^n\right) = 0.$$

collecting similar terms we get:

$$(x^2y_1'' + xy_1' + x^2y_1)\ln x + 2xy_1' + \sum_{n=2}^{\infty} n(n-1)c_n x^n$$
$$+ \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=1}^{\infty} c_n x^{n+2} = 0.$$

But y_1 is a solution to the equation: $x^2y_1'' + xy_1' + x^2y_1 = 0$, so we get

$$2xy_1' + \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=1}^{\infty} c_n x^{n+2} = 0.$$

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

$$y_1' = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2}$$

$$2xy_1' = 2\sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n}}{2^{2n}(n!)^2}$$

$$2\sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n}}{2^{2n}(n!)^2} + \sum_{n=2}^{\infty}(n(n-1)+n)c_n x^n + c_1 x + \sum_{n=3}^{\infty} c_{n-2} x^n = 0$$

$$2\sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n}}{2^{2n}(n!)^2} + c_1 x + 2^2 c_2 x^2 + \sum_{n=3}^{\infty}(n^2 c_n + c_{n-2})x^n = 0. \quad (*)$$

The only term involving x in this equation is $c_1 x$ so $c_1 = 0$.

All odd powers come from: $\sum_{n=3}^{\infty}(n^2 c_n + c_{n-2})x^n$,

so $n^2 c_n + c_{n-2} = 0,\ n$ odd.

$c_n = -\frac{c_{n-2}}{n^2}$ for n odd.

But since $c_1 = 0$, all the odd coefficients are 0.

From equation $(*)$ we can calculate the even coefficients, c_{2n}.

$n = 1$:

$$2\left(\frac{(-1)(2)}{4}\right) + 2^2 c_2 = 0$$

$$c_2 = \frac{4}{4}\left(\frac{1}{2^2}\right) = \frac{1}{4}$$

$n \geq 2$:

$$\frac{2(-1)^n 2n}{2^{2n}(n!)^2} + (2n)^2 c_{2n} + c_{2n-2} = 0$$

$$4n^2 c_{2n} + c_{2n-2} = -\frac{2(-1)^n 2n}{2^{2n}(n!)^2}$$

$$c_{2n} = \frac{1}{4n^2}\left[-c_{2n-2} + \frac{2(-1)^{n+1} 2n}{2^{2n}(n!)^2}\right]$$

$n = 2$:

$$c_4 = \frac{1}{4(2)^2}\left[-c_2 - \frac{2}{4(4)}\right]$$

$$c_4 = \frac{1}{16}\left[-\frac{1}{4} - \frac{1}{8}\right] = -\frac{3}{128}$$

$n = 3$:

$$c_6 = \frac{1}{4(3)^2}\left[-c_4 + \frac{3}{2^4(3!)^2}\right]$$

$$c_6 = \frac{1}{36}\left[\frac{3}{128} + \frac{3}{16(36)}\right] = \frac{11}{13824}\,.$$

So we get:

$$y_2(x) = \left(J_0(x)\right)\ln x + \frac{x^2}{4} - \frac{3}{128}x^4 + \frac{11}{13824}x^6 + \cdots.$$