Series Solutions Near Regular Singular Points: The Frobenius Method

If $A(x)y'' + B(x)y' + C(x)y = 0$ and A, B, and C have no common factors then points where $A(x) = 0$ are singular points of this equation.

Ex. $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ (Legendre's Equation) has singular points at $x = \pm 1$.

We will focus our attention on situations where $x = 0$ is the singular point. If $x = a$ were a singular point we could always make a substitution, $t = x - a$, which would have a singular point at $t = 0$.

We will consider equations of the form: $A(x)y'' + B(x)y' + C(x)y = 0$, where A, B, and C are analytic at $x = 0$ (i.e. $A(x), B(x)$, and $C(x)$ have convergent power series in x around $x = 0$).

In general, if $A(x) = 0$ at $x = 0$, we will not be able to solve the equation with a power series. However, in certain circumstances we will be able to generalize the power series approach.

Ex. Bessel's Equation has a singularity at $x = 0$.

$$x^2y'' + xy' + x^2y = 0$$

or

$$y'' + \frac{1}{x}y' + y = 0.$$

If we take $A(x)y'' + B(x)y' + C(x)y = 0$ and divide by $A(x)$ we get:

$$y'' + P(x)y' + Q(x)y = 0$$

where $P(x) = \frac{B(x)}{A(x)}$, $Q(x) = \frac{C(x)}{A(x)}$.

In our example, $P(x) = \frac{1}{x}$, $Q(x) = 1$.

We will see that we will be able to generalize the power series approach if $P(x)$ approaches infinity no more rapidly than $\frac{1}{x}$ and $Q(x)$ approaches infinity no more rapidly than $\frac{1}{x^2}$ as x goes to zero from the right.

If we rewrite $y'' + P(x)y' + Q(x)y = 0$ in the form:

$$y'' + \frac{p(x)}{x}y' + \frac{q(x)}{x^2}y = 0$$

where $p(x) = xP(x)$ and $q(x) = x^2Q(x)$, then we have the following definition.

Def. The singular point $x = 0$ is a **regular singular point** if the functions $p(x)$ and $q(x)$ are both analytic at $x = 0$. Otherwise, it is an **irregular singular point**.

Ex. $$x^2y'' + xy' + x^2y = 0$$

$$y'' + \frac{1}{x}y' + y = 0.$$

Here, $p(x) = x\left(\frac{1}{x}\right) = 1$ and $q(x) = x^2(1) = x^2$.

Both $p(x)$ and $q(x)$ are analytic at $x = 0$ so $x = 0$ is a regular singular point for this equation.

Ex. Consider the equation:

$$3x^4y'' + x^2y' - (2x^3 + x^2)y = 0.$$

Then:

$$y'' + \frac{1}{3x^2}y' - \frac{2x^3+x^2}{3x^4}y = 0$$

or:

$$y'' + \frac{\frac{1}{3x}}{x}y' - \frac{\frac{2x+1}{3}}{x^2}y = 0.$$

So for this equation:

$$p(x) = \frac{1}{3x} \text{ and } q(x) = \frac{2x+1}{3}.$$

In this case, $p(x)$ is not analytic at $x = 0$ since $\lim_{x \to 0^+} \frac{1}{3x} = +\infty$.

Thus, $x = 0$ is not a regular singular point.

Ex. Determine if $x = 0$ is an ordinary point, a regular singular point, or an irregular singular point for the following equation:

$$x^2(3 - x)y'' + (5x + x^3)y' + (2x - 3)y = 0.$$

$x = 0$ is a singular point since $x^2(3 - x) = 0$ and $(2x - 3) \neq 0$ at $x = 0$. So

$$y'' + \frac{x(5+x^2)}{x^2(3-x)}y' + \frac{2x-3}{x^2(3-x)}y = 0$$

$$y'' + \frac{5+x^2}{x(3-x)}y' + \frac{2x-3}{x^2(3-x)}y = 0$$

$$y'' + \frac{\frac{5+x^2}{(3-x)}}{x}y' + \frac{\frac{2x-3}{(3-x)}}{x^2}y = 0$$

$p(x) = \frac{5+x^2}{3-x}$ and $q(x) = \frac{2x-3}{3-x}$ are both analytic at $x = 0$ since they are rational functions where denominators are not zero at $x = 0$.

Thus, $x = 0$ is a regular singular point.

The Method of Frobenius

Suppose we want to solve,

$$x^2y'' + \frac{5}{2}xy' - y = 0.$$

Let's guess the solution is $y = x^r$.

$$y = x^r$$
$$y' = rx^{r-1}$$
$$y'' = r(r-1)x^{r-2}.$$

Now substitute into the differential equation:

$$x^2r(r-1)x^{r-2} + \frac{5}{2}x(r)x^{r-1} - x^r = 0$$
$$r(r-1)x^r + \frac{5}{2}rx^r - x^r = 0$$
$$\left(r(r-1) + \frac{5}{2}r - 1\right)x^r = 0$$
$$\left(r - \frac{1}{2}\right)(r+2)x^r = 0$$
$$r = -2\,,\frac{1}{2}\,.$$

So $y = x^{\frac{1}{2}}$ or $y = x^{-2}$ are solutions.

Notice that even though all of coefficients of the original equation are analytic at $x = 0$, the solutions are not.

In general, if we have to solve $x^2y'' + xp(x)y' + q(x)y = 0$, where $p(x)$ and $q(x)$ are power series, we might guess the solution has the form:

$$y = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r} = c_0x^r + c_1x^{r+1} + \cdots.$$

An infinite series of this form is called a **Frobenius series**. Notice that it need not be a power series as r may not be a positive integer. For example, if $r = -\frac{1}{2}$ then:

$$y = c_0 x^{-\frac{1}{2}} + c_1 x^{\frac{1}{2}} + c_2 x^{\frac{3}{2}} + \cdots.$$

Ex. Let's solve $x^2 y'' + xp(x)y' + q(x)y = 0$, where:

$$p(x) = p_0 + p_1 x + p_2 x^2 + \cdots$$
$$q(x) = q_0 + q_1 x + q_2 x^2 + \cdots\ .$$

Let's assume: $y = \sum_{n=0}^{\infty} c_n x^{n+r}$

where $c_0 \neq 0$ (the series must have some nonzero first term)

$$y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}.$$

Substituting into $x^2 y'' + xp(x)y' + q(x)y = 0$, we get:

$$\begin{aligned} x^2(r(r-1)c_0 x^{r-2} + (r+1)(r)c_1 x^{r-1} + \cdots) \\ + x(p_0 + p_1 x + p_2 x^2 + \cdots)(c_0 r x^{r-1} + c_1(r+1)x^r + \cdots) \\ + (q_0 + q_1 x + q_2 x^2 + \cdots)(c_0 x^r + c_1 x^{r+1} + \cdots) = 0 \end{aligned}$$

$$\begin{aligned} \big((r)(r-1)c_0 x^r + (r+1)(r)c_1 x^{r+1} + \cdots\big) \\ + (p_0 + p_1 x + \cdots)(c_0 r x^r + c_1(r+1)x^{r+1} + \cdots) \\ +(q_0 + q_1 x + \cdots)(c_0 x^r + c_1 x^{r+1} + \cdots) = 0. \end{aligned}$$

Let's collect the coefficients of the x^r term and set the expression equal to 0.

$$(r(r-1)c_0 + p_0c_0r + q_0c_0)x^r = 0$$

$$c_0[(r)(r-1) + p_0r + q_0] = 0$$

$$r(r-1) + p_0r + q_0 = 0, \quad \text{since } c_0 \neq 0.$$

This last equation is called the **indicial equation** of the differential equation and the roots, r, are the **exponents** of the differential equation. If $r_1 \neq r_2$, then there are two possible Frobenius series solutions. If $r_1 = r_2$, then there is only one solution, and the second one can't be found with this method. Notice, p_0 and q_0 in the indicial equation, $r(r-1) + p_0r + q_0 = 0$, are just the values of $p(x)$ and $q(x)$ at $x = 0$.

Ex. Find the exponents and the possible Frobenius series solutions of:

$$x^2(1-x^2)y'' + 2xy' - 2y = 0.$$

Dividing by $x^2(1-x^2)$ we get:

$$y'' + \frac{2}{x(1-x^2)}y' - \frac{2}{x^2(1-x^2)}y = 0$$

$$y'' + \frac{\frac{2}{1-x^2}}{x}y' - \frac{\frac{2}{1-x^2}}{x^2}y = 0.$$

So $\quad p(x) = \frac{2}{1-x^2}$ and $p(0) = 2$

$\quad q(x) = -\frac{2}{1-x^2}$ and $q(0) = -2$.

$$r(r-1)+p_0r+q_0=r(r-1)+2r-2=0$$

$$r^2-r+2r-2=r^2+r-2=0$$

$$(r+2)(r-1)=0$$

$$r=-2,\ 1.$$

So the two possible Frobenius solutions are:

$$y_1(x)=x^{-2}\sum_{n=0}^{\infty}a_nx^n\,; \qquad y_2(x)=x\sum_{n=0}^{\infty}b_nx^n\,.$$

Theorem: Suppose $x=0$ is a regular singular point of

$x^2y''+xp(x)y'+q(x)y=0$. Let $\rho>0$ denote the minimum of the radii of convergence of the power series

$p(x)=\sum_{n=0}^{\infty}p_nx^n$ and $q(x)=\sum_{n=0}^{\infty}q_nx^n$.

Let r_1 and r_2 be the real roots, with $r_1\geq r_2$, of the indicial equation:

$r(r-1)+p_0r+q_0=0$. Then we can say,

a) For $x>0$ there exists a solution of the equation $x^2y''+xp(x)y'+q(x)y=0$, of the form:

$y_1(x)=x^{r_1}\sum_{n=0}^{\infty}a_nx^n\ ,\ a_0\neq 0$ corresponding to the larger root (r_1)

b) If r_1-r_2 is neither 0 nor a positive integer, then there exists a second linearly independent solution for $x>0$ of the form:

$y_2(x)=x^{r_2}\sum_{n=0}^{\infty}b_nx^n\ ,\ b_0\neq 0$ corresponding to the smaller root (r_2)

The radii of convergence of the power series for $y_1(x)$ and $y_2(x)$ are at least ρ. The coefficients in these series can be determined by substituting the series in the differential equation.

Ex. Find the Frobenius series solutions of:

$$2x^2y'' + xy' - (2x^2+1)y = 0.$$

Dividing by $2x^2$ we get,

$$y'' + \frac{\frac{1}{2}}{x}y' - \frac{\frac{1}{2}(2x^2+1)}{x^2}y = 0.$$

So $x = 0$ is a regular singular point and $p_0 = \frac{1}{2}, \quad q_0 = -\frac{1}{2}$ because $p(x) = \frac{1}{2}, \quad q(x) = -\frac{1}{2}(2x^2+1)$. Thus the indicial equation becomes:

$$r(r-1) + \frac{1}{2}r - \frac{1}{2} = 0$$

$$r^2 - r + \frac{1}{2}r - \frac{1}{2} = 0$$

$$r^2 - \frac{1}{2}r - \frac{1}{2} = 0$$

$$\left(r + \frac{1}{2}\right)(r-1) = 0$$

$$r = -\frac{1}{2}, \; 1$$

$r_1 - r_2$ is neither zero nor a positive integer so we should get two Frobenius solutions:

$$y_1(x) = x^{-\frac{1}{2}} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x \sum_{n=0}^{\infty} b_n x^n.$$

We will substitute $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and solve for the coefficients in terms of r. Then we will substitute $r_1 = -\frac{1}{2}$ and $r_2 = 1$:

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$
$$y' = \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}$$

$$2x^2 \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + x\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} -(2x^2+1)\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} - \sum_{n=0}^{\infty} 2c_n x^{n+r+2} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

Notice that: $\sum_{n=0}^{\infty} 2c_n x^{n+r+2} = \sum_{n=2}^{\infty} 2c_{n-2} x^{n+r}$.

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} - \sum_{n=2}^{\infty} 2c_{n-2} x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

The common range of the summations is $n \geq 2$, so we have to handle $n = 0$, $n = 1$ separately.

$$n = 0: \quad [2r(r-1) + r - 1]c_0 = 2\left(r^2 - \frac{1}{2}r - \frac{1}{2}\right)c_0 = 0$$

Notice we get the indicial equation. This will always happen for $n = 0$.

Since $r^2 - \frac{1}{2}r - \frac{1}{2} = 0$ both r_1 and r_2, c_0 is an arbitrary constant.

$n = 1$: $[2(1+r)(r) + (1+r) - 1]c_1 = (2r^2 + 3r)c_1 = 0.$

Since $2r^2 + 3r \neq 0$ for $r = -\frac{1}{2}$ or 1, c_1 must be 0 in either case.

The coefficient of x^{n+r} for $n \geq 2$ is:

$$2(n+r)(n+r-1)c_n + (n+r)c_n - 2c_{n-2} - c_n = 0$$

$$[2(n+r)(n+r-1) + (n+r) - 1]c_n = 2c_{n-2}$$

$$[2(n+r)^2 - (n+r) - 1]c_n = 2c_{n-2}$$

$$c_n = \frac{2c_{n-2}}{2(n+r)^2 - (n+r) - 1} \quad ; \quad n \geq 2.$$

Case 1: $r_1 = -\frac{1}{2}$

$$a_n = \frac{2a_{n-2}}{2\left(n-\frac{1}{2}\right)^2 - \left(n-\frac{1}{2}\right) - 1} = \frac{2a_{n-2}}{2n^2-3n} = \frac{2a_{n-2}}{n(2n-3)}; \qquad n \geq 2$$

Since $c_1 = a_1 = 0$, $a_{2n+1} = 0$ for all n. Let's look at $n = 2, 4, 6, 8$:

$n = 2$ $\qquad a_2 = a_0$

$n = 4$ $\qquad a_4 = \frac{a_2}{2(5)} = \frac{a_0}{2(5)}$

$n = 6$ $\qquad a_6 = \frac{a_4}{3(9)} = \frac{a_0}{2(3)(5)(9)}$

$n = 8$ $\qquad a_8 = \frac{a_6}{4(13)} = \frac{a_0}{2(3)(4)(5)(9)(13)}$

$$a_{2n} = \frac{a_0}{n!(5)(9)\cdots(4n-3)}$$

$$y_1(x) = a_0 x^{-\frac{1}{2}}\left(1 + x^2 + \frac{1}{2(5)}x^4 + \frac{1}{2(3)(5)(9)}x^6 + \cdots + \frac{1}{n!(5)(9)\cdots(4n-3)}x^{2n} + \cdots\right).$$

$$y_1(x) = a_0 x^{-\left(\frac{1}{2}\right)}(1 + \sum_{n=1}^{\infty} \frac{1}{n!(5)(9)\cdots(4n-3)} x^{2n}).$$

Case 2: $r_2 = 1$

$$b_n = \frac{2b_{n-2}}{2(n+1)^2-(n+1)-1} = \frac{2b_{n-2}}{2n^2+3n} = \frac{2b_{n-2}}{n(2n+3)}; \qquad n \geq 2.$$

Once again $c_1 = 0$ implies all odd b_ns are 0. For $n = 2, 4, 6$:

$n = 2$ $\qquad b_2 = \frac{b_0}{7}$

$n = 4$ $\qquad b_4 = \frac{b_2}{2(11)} = \frac{b_0}{2(7)(11)}$

$n = 6$ $\qquad b_6 = \frac{b_4}{3(15)} = \frac{b_0}{2(3)(7)(11)(15)}$

$$b_{2n} = \frac{b_0}{n!(7)(11)(15)\cdots(4n+3)}$$

$$y_2(x) = b_0 x(1 + \frac{x^2}{7} + \frac{x^4}{2(7)(11)} + \frac{x^6}{2(3)(7)(11)(15)} + \cdots + \frac{x^{2n}}{n!(7)(11)(15)\cdots(4n+3)} + \cdots)$$

$$y_2(x) = b_0 x(1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n!(7)(11)(15)\cdots(4n+3)}).$$

General Solution:

$$y(x) = a_0 x^{-\left(\frac{1}{2}\right)}\left(1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n!\,(5)(9)\cdots(4n-3)}\right)$$

$$+b_0 x(1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n!\,(7)(11)\cdots(4n+3)})$$

Ex. Find a Frobenius solution to Bessel's equation of order 0:

$$x^2y'' + xy' + x^2y = 0.$$

Dividing by x^2 we get: $y'' + \frac{1}{x}y' + \frac{x^2}{x^2}y = 0$

So $p(x) = 1,\ q(x) = x^2$ and $p_0 = 1,\ q_0 = 0.$

The indicial equation becomes: $r(r-1) + r + 0 = 0$

$$r^2 = 0; \quad \text{so } r = 0.$$

So there is only one Frobenius series solution since $r_1 - r_2 = 0.$

$$y(x) = x^0 \sum_{n=0}^{\infty} c_n x^n.$$

This is, in fact, a power series. Substituting we get:

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=0}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$$

$$x^2\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=0}^{\infty} nc_n x^{n-1} + x^2\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)c_n x^n + \sum_{n=0}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^{n+2} = 0.$$

Notice that: $\sum_{n=0}^{\infty} c_n x^{n+2} = \sum_{n=2}^{\infty} c_{n-2} x^n$.

$$\sum_{n=0}^{\infty} n(n-1)c_n x^n + \sum_{n=0}^{\infty} nc_n x^n + \sum_{n=2}^{\infty} c_{n-2} x^n = 0$$

$$\sum_{n=0}^{\infty}[n(n-1)+n]c_n x^n + \sum_{n=2}^{\infty} c_{n-2} x^n = 0$$

$$\sum_{n=0}^{\infty} n^2 c_n x^n + \sum_{n=2}^{\infty} c_{n-2} x^n = 0.$$

The first series starts with $n = 0$ and the second with $n = 2$ so we need to separate the first two terms of the first series:

$$0^2(c_0) + (1)^2(c_1)x + \sum_{n=2}^{\infty} n^2 c_n x^n + \sum_{n=2}^{\infty} c_{n-2} x^n = 0$$

$$0^2(c_0) + (1)^2(c_1)x + \sum_{n=2}^{\infty}[n^2 c_n + c_{n-2}]x^n = 0.$$

For $n = 0$ we get $0^2(c_0) = 0$, so c_0 can be any constant.

For $n = 1$ we get $(1)^2(c_1) = 0$, so $c_1 = 0$.

For $n \geq 2$ we get $n^2 c_n + c_{n-2} = 0$, so $c_n = -\frac{1}{n^2}c_{n-2}$.

For $n = 2, 4, 6$ we get:

$$c_2 = -\frac{1}{2^2} c_0$$

$$c_4 = -\frac{1}{4^2} c_2 = \frac{1}{2^2(4^2)} c_0$$

$$c_6 = -\frac{1}{6^2} c_4 = -\frac{1}{2^2(4^2)(6^2)} c_0$$

$$\Longrightarrow \quad c_{2n} = \frac{(-1)^n}{(2^2)(4^2)(6^2)\ldots(2n)^2} c_0.$$

Since $(2^2)(4^2)(6^2) \ldots (2n)^2 = 2^{2n}(1^2)(2^2) \ldots (n)^2 = 2^{2n}(n!)^2$

$$y_1(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

If we let $c_0 = 1$ we get the **Bessel function of order zero of the first kind**:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

We were only able to find one solution to $x^2y'' + xy' - x^2y = 0.$

Later we will find a second linearly independent solution (which is not a Frobenius series).