Series Solutions Near Ordinary Points

A general homogeneous second order linear differential equation with analytic coefficients (i.e. the functions can be represented by power series) has the form:

$$A(x)y'' + B(x)y' + C(x)y = 0.$$

If $A(x) \neq 0$ for any x in an open interval, I, then we can divide the equation by $A(x)$ to get:

$$y'' + P(x)y' + Q(x)y = 0.$$

Here, $P(x) = \frac{B(x)}{A(x)}$ and $Q(x) = \frac{C(x)}{A(x)}$, but what happens if there is a point where $A(x) = 0$?

Ex. $x^2y'' + y' + x^3y = 0$, dividing by x^2 we get:

$$y'' + \frac{1}{x^2}y' + xy = 0.$$

$P(x) = \frac{1}{x^2}$ is not analytic at $x = 0$.

Def. If we consider the equation: $y'' + P(x)y' + Q(x)y = 0$, then $x = a$ is called an **ordinary point** of this equation (and of the equation $A(x)y'' + B(x)y' + C(x)y = 0$) if $P(x)$ and $Q(x)$ are both analytic at $x = a$. Otherwise $x = a$ is called a **singular point**.

Ex. For the equation $y'' + \frac{1}{x^2}y' + xy = 0$, $x = 0$ is a singular point and $x \neq 0$ are ordinary points.

Ex. For the equation $(x^2-1)y''+(3x^2+1)y'+(5-x)y=0$,
$x=0$ is an ordinary point but $x=\pm 1$ are singular points because:

$$y''+\frac{(3x^2+1)}{(x^2-1)}y'+\frac{(5-x)}{(x^2-1)}y=0$$

$P(x)$ and $Q(x)$ don't have convergent Taylor series around $x=\pm 1$, but they do for $x\neq\pm 1$ (which includes $x=0$).

So $x\neq\pm 1$ are all ordinary points.

Theorem: Suppose that a is an ordinary point of the equation,

$$A(x)y''+B(x)y'+C(x)y=0.$$

That is, $P(x)=\frac{B(x)}{A(x)}$ and $Q(x)=\frac{C(x)}{A(x)}$ are analytic at $x=a$.
Then the differential equation has two linearly independent solutions each of the form:

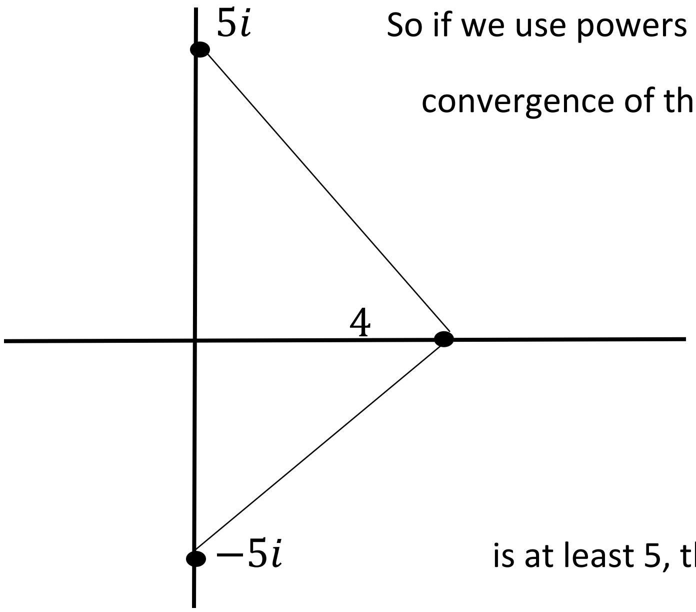
$$y(x)=\sum_{n=0}^{\infty}c_n(x-a)^n.$$

The radius of convergence is at least as large as the distance of a to the nearest (real or complex) singular point.

Ex. Determine the radius of convergence guaranteed by the previous theorem of a series solution of: $(x^2+25)y''+xy'+x^2y=0$ in powers of x. What if it's in powers of $x-4$?

$$P(x)=\frac{x}{x^2+25}\ ;\quad Q(x)=\frac{x^2}{x^2+25}$$

$P(x)$ and $Q(x)$ have singular points at $x=\pm 5i$.



So if we use powers of $x, a = 0$ and the radius of convergence of the series solution: $y = \sum_{n=0}^{\infty} c_n x^n$.

is at least 5, the distance between 0 and the nearest singular point $\pm 5i$.

If the series solution is powers of $x - 4$, then $a = 4$. The distance between 4 and $\pm 5i$ is $\sqrt{41}$. So the radius of convergence of a series solution $y = \sum_{n=0}^{\infty} d_n (x-4)^n$ is at least $\sqrt{41}$.

Ex. Find the general solution in powers of x of

$$(x^2 - 2)y'' + 5xy' + 4y = 0.$$

Then find the particular solution with $y(0) = 4,\ y'(0) = 1$.

The only singular points are at $x = \pm\sqrt{2}$, so the radius of convergence of the solution should be at least $\sqrt{2}$.

$$y = \sum_{n=0}^{\infty} c_n x^n\,; \quad y' = \sum_{n=1}^{\infty} nc_n x^{n-1}\,; \quad y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

Substituting into $(x^2 - 2)y'' + 5xy' + 4y = 0$ we get:

$$(x^2 - 2)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 5x\sum_{n=1}^{\infty} nc_n x^{n-1} + 4\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=2}^{\infty} 2n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} 5nc_n x^n + \sum_{n=0}^{\infty} 4c_n x^n = 0.$$

Notice we can start the first and third sums from $n = 0$ since that won't change the values of each expression.

$$\sum_{n=0}^{\infty} n(n-1)c_n x^n - \sum_{n=2}^{\infty} 2n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} 5nc_n x^n + \sum_{n=0}^{\infty} 4c_n x^n = 0.$$

The powers of x in the second term don't line up with the others so we can say:

$$\sum_{n=2}^{\infty} 2n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2)(n+1)c_{n+2} x^n$$

$$\sum_{n=0}^{\infty} n(n-1)c_n x^n - \sum_{n=0}^{\infty} 2(n+2)(n+1)c_{n+2} x^n + \sum_{n=0}^{\infty} 5nc_n x^n + \sum_{n=0}^{\infty} 4c_n x^n = 0$$

$$\sum_{n=0}^{\infty} [n(n-1)c_n - 2(n+1)(n+2)c_{n+2} + 5nc_n + 4c_n]x^n = 0.$$

So solve: $n(n-1)c_n - 2(n+1)(n+2)c_{n+2} + 5nc_n + 4c_n = 0$

$$[n(n-1) + 5n + 4]c_n = 2(n+1)(n+2)c_{n+2}$$

$$(n^2 + 4n + 4)c_n = 2(n+1)(n+2)c_{n+2}$$

$$c_{n+2} = \frac{n+2}{2(n+1)}c_n \quad \text{for } n \geq 0.$$

With $n = 0, 2, 4$ we get

$$c_2 = \frac{2}{2(1)}c_0$$

$$c_4 = \frac{4}{2(3)}c_2 = \frac{2(4)}{2^2(1)(3)}c_0$$

$$c_6 = \frac{6}{2(5)}c_4 = \frac{2(4)(6)}{2^3(1)(3)(5)}c_0$$

$$\Longrightarrow \qquad c_{2n} = \frac{2(4)(6)\cdots(2n)}{2^n(1)(3)(5)\cdots(2n-1)}c_0; \qquad n \geq 1.$$

Let's define $(2n+1)!! = 1(3)(5)\cdots(2n+1) = \frac{(2n+1)!}{2^n(n!)}$,

and we know $2(4)(6)\cdots(2n) = 2^n(n!)$, so we can say:

$$c_{2n} = \frac{2^n(n!)}{2^n(2n-1)!!}c_0 = \frac{n!}{(2n-1)!!}c_0.$$

With $n = 1, 3, 5$ we get

$$c_3 = \frac{3}{2(2)} c_1$$

$$c_5 = \frac{5}{2(4)} c_3 = \frac{3(5)}{2^2(2)(4)} c_1$$

$$c_7 = \frac{7}{2(6)} c_5 = \frac{3(5)(7)}{2^3(2)(4)(6)} c_1$$

$$c_{2n+1} = \frac{3(5)(7)\cdots(2n+1)}{2^n(2)(4)(6)\cdots(2n)} c_1 = \frac{(2n+1)!!}{2^n(2^n)(n!)} c_1 = \frac{(2n+1)!!}{2^{2n}(n!)} c_1, \quad n \geq 1.$$

So we now have the general solution:

$$y(x) = c_0\left(1 + \sum_{n=1}^{\infty} \frac{n!}{(2n-1)!!} x^{2n}\right) + c_1\left(x + \sum_{n=1}^{\infty} \frac{(2n+1)!!}{2^{2n}(n!)} x^{2n+1}\right)$$

or alternatively:

$$y(x) = c_0\left(1 + x^2 + \frac{2}{3}x^4 + \frac{2}{5}x^6 + \cdots\right) + c_1\left(x + \frac{3}{4}x^3 + \frac{15}{32}x^5 + \frac{35}{128}x^7 + \cdots\right).$$

Notice that $y(0) = c_0, \;\; y'(0) = c_1$ so we can write:

$$4 = y(0) = c_0\,, \;\; 1 = y'(0) = c_1 \qquad \Longrightarrow$$

$$y(x) = 4\left(1 + \sum_{n=1}^{\infty} \frac{n!}{(2n-1)!!} x^{2n}\right) + \left(x + \sum_{n=1}^{\infty} \frac{(2n+1)!!}{2^{2n}(n!)} x^{2n+1}\right)$$

or alternatively:

$$y(x) = 4 + x + 4x^2 + \frac{3}{4}x^3 + \frac{8}{3}x^4 + \frac{15}{32}x^5 + \cdots.$$

Translated Series Solutions

If in the previous example we were given initial conditions in terms of $y(a)$ and $y'(a)$, where $a \neq 0$, we would need to find the general solution as a power series around a:

$$y(x) = \sum_{n=0}^{\infty} c_n (x-a)^n.$$

One way around this is to make a substitution, $z = x - a$, and get a solution in terms of z:

$$y(z) = \sum_{n=0}^{\infty} c_n z^n$$

Ex. Solve the initial value problem:

$$(t^2 - 2t - 1)\frac{d^2y}{dt^2} + 5(t-1)\frac{dy}{dt} + 4y = 0$$

$$y(1) = 4\,, \qquad y'(1) = 1.$$

We want a solution: $\quad y = \sum_{n=0}^{\infty} c_n (t-1)^n.$

Make a substitution $x = t - 1$ into the differential equation (we can do this since $x = t - 1 \implies x + 1 = t$).

$$t^2 - 2t - 1 = (x+1)^2 - 2(x+1) - 1 = x^2 - 2$$

$$5(t-1) = 5x$$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx} = y'$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \left(\frac{d}{dx}(y')\right)\frac{dx}{dt} = y''$$

So the differential equation becomes:

$$(x^2 - 2)y'' + 5xy' + 4y = 0$$

$$y(0) = 4\,, \qquad y'(0) = 1$$

Which we solved in the previous example.

So the solution to the original problem is what we get when we substitute $x = t - 1$ in our previous solution:

$$y(t) = 4 + (t-1) + 4(t-1)^2 + \tfrac{3}{4}(t-1)^3 + \tfrac{8}{3}(t-1)^4 + \cdots$$

Legendre's Equation

The Legendre Equation of order α is:

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$$

This equation comes up in many places. Among them is the problem of determining the steady state temperature within a solid sphere when the temperature on the boundary is known.

If we substitute a power series in x for y'', y', and y and combine the coefficients of x^n we will find:

$$c_{m+2} = \frac{(\alpha-m)(\alpha+m+1)}{(m+1)(m+2)}c_m.$$

This leads to:

$$c_{2m} = (-1)^m \frac{\alpha(\alpha-2)(\alpha-4)\ldots(\alpha-2m+2)(\alpha+1)(\alpha+3)\ldots(\alpha+2m-1)}{(2m)!} c_0$$

$$c_{2m+1} = (-1)^m \frac{(\alpha-1)(\alpha-3)\ldots(\alpha-2m+1)(\alpha+2)(\alpha+4)\ldots(\alpha+2m)}{(2m+1)!} c_1.$$

We can then write:

$$c_{2m} = (-1)^m a_{2m} c_0 \quad \text{and} \quad c_{2m+1} = (-1)^m a_{2m+1} c_1$$

where a_{2m} and a_{2m+1} are the messy fractions in the expressions of c_{2m} and c_{2m+1}.

The general solution to Legendre's Equation then becomes:

$$y(x) = c_0 \sum_{m=0}^{\infty} (-1)^m a_{2m} x^{2m} + c_1 \sum_{m=0}^{\infty} (-1)^m a_{2m+1} x^{2m+1}.$$

Now notice that if $\alpha = n$, a positive even integer, then $a_{2m} = 0$ when $2m > n$. In that case:

$$c_0 \sum_{m=0}^{\infty} (-1)^m a_{2m} x^{2m}$$

is a polynomial of degree n (i.e. there are a finite number of terms) but,

$$c_1 \sum_{m=0}^{\infty} (-1)^m a_{2m+1} x^{2m+1}$$

is a nonterminating infinite series (it has an infinite number of nonzero terms).

If $\alpha = n$, a positive odd integer, then $a_{2m+1} = 0$ when $2m + 1 > n$. In this case:

$$c_1 \sum_{m=0}^{\infty} (-1)^m \, a_{2m+1} x^{2m+1}$$

is a polynomial of degree n and,

$$c_0 \sum_{m=0}^{\infty} (-1)^m \, a_{2m} x^{2m}$$

is a nonterminating infinite series.

So if n is an integer, the n^{th} degree polynomial solution of:

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

is denoted by $P_n(x)$ and is called the Legendre Polynomial of degree n.