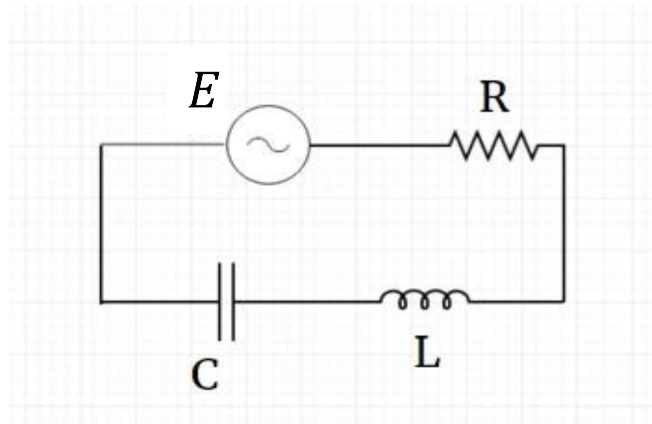
Electrical Circuits

An RLC circuit consists of:

A resistor with a resistance of R ohms

An inductor with an inductance of L henries

A capacitor with a capacitance of C farads



We assume that there is a source (a battery or generator) that supplies a voltage of $E(t)$.

If the circuit is closed, a current of $I(t)$ amperes is generated and a charge of $Q(t)$ coulombs on the capacitor at time t.

We have the following relation:

$$\frac{dQ}{dt} = I(t).$$

The voltage drop across the three circuit elements is given by:

Circuit Element	Voltage Drop
Inductor	$L\frac{dI}{dt}$
Resistor	RI
Capacitor	$\frac{1}{C}Q$

This leads to the following relationship:

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t).$$

Substituting $\frac{dQ}{dt} = I$, we get:

$$LQ'' + RQ' + \frac{1}{C}Q = E(t).$$

In many practical problems it's the current, I, that is of interest rather than the charge Q. Differentiating the previous equation we get:

$$LQ''' + RQ'' + \frac{1}{C}Q' = E'(t)$$

$$LI'' + RI' + \frac{1}{C}I = E'(t).$$

Notice that just like the case for a mechanical system, we have a second order linear differential equation with constant (and positive) coefficients.

Mechanical System	Electrical System
Mass m	Inductance L
Damping constant c	Resistance R
Spring constant k	Reciprocal of Capacitance $\frac{1}{C}$
Position x	Charge Q or current I
Force F	Electromotive force E or $E'(t)$.

Ex. If we have a circuit with no capacitor we have:

$$LI' + RI = E(t).$$

Suppose $L = 6\ H,\ R = 36$ ohms, and an alternating current generator supplies a voltage of $E(t) = 150\cos(50t)$. Given $I(0) = 0$, Find $I(t)$.

$$6I' + 36I = 150\cos(50t)$$

$$I' + 6I = 25\cos(50t).$$

First, solve the homogeneous equation:

$$I' + 6I = 0$$

$$\frac{dI}{dt} = -6I$$

$$\frac{dI}{I} = -6\ dt$$

$$\int \frac{dI}{I} = \int -6\ dt$$

$$\ln I = -6t + c$$

$$I = e^{-6t+c} = c_1 e^{-6t}$$

$$I_c = c_1 e^{-6t}.$$

Now find I_p. We guess the particular solution is of the form:

$$I_p = A\cos(50t) + B\sin(50t)$$

$$I'_p = -50\,\mathrm{A}\sin(50t) + 50B\cos(50t).$$

Now substituting in $I' + 6I = 25\cos(50t)$ we get:

$$-50A\sin(50t) + 50B\cos(50t) + 6(A\cos(50t) + B\sin(50t)) = 25\cos(50t)$$

$$(-50A + 6B)\sin(50t) + (50B + 6A)\cos(50t) = 25\cos(50t).$$

$$-50A + 6B \ = 0 \qquad \Rightarrow B = \frac{50}{6}A = \frac{25}{3}A$$

$$50B + 6A = 25$$ Now substitute for B.

$$50(\frac{25}{3}A) + 6A = 25$$

$$\frac{1268}{3}A = 25$$

$$\Longrightarrow \qquad A = \frac{75}{1268}, \qquad B = \frac{25}{3}\left(\frac{75}{1268}\right) = \frac{625}{1268}.$$

$$\Longrightarrow \quad I_p = \frac{75}{1268}\cos(50t) + \frac{625}{1268}\sin(50t).$$

$$I(t) = I_c + I_p = c_1 e^{-6t} + \frac{75}{1268}\cos(50t) + \frac{625}{1268}\sin(\,50t).$$

$$0 = I(0) = c_1 + \frac{75}{1268}$$

$$c_1 = -\frac{75}{1268}.$$

$$I(t) = -\frac{75}{1268}e^{-6t} + \frac{75}{1268}\cos 50t + \frac{625}{1268}\sin 50t.$$

Ex. Now assume we have an RC circuit $(L = 0)$. Then we get:

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t).$$

Suppose $R = 100,\ C = 10^{-4},\ Q(0) = 0$, and $E(t) = 200\cos(50t)$.

Find $Q(t)$ and $I(t)$.

$$100\frac{dQ}{dt} + \frac{1}{10^{-4}}Q = 200\cos(50t)$$

$$Q' + 100Q = 2\cos 50t.$$

Solve the homogeneous equation:

$$Q' + 100Q = 0$$

$$\frac{dQ}{dt} = -100Q$$

$$\frac{dQ}{Q} = -100\ dt$$

$$\int \frac{dQ}{Q} = \int -100\ dt$$

$$\ln Q = -100\ t + c$$

$$Q_c(t) = e^{(-100t+c)} = c_1 e^{(-100t)}.$$

We guess that Q_p is of the form:

$$Q_p = A\cos(50t) + B\sin(50t)$$

$$Q'_p = -50\,\mathrm{A}\sin 50t + 50B\cos 50t.$$

Substituting into $Q' + 100Q = 2\cos(50t)$ we get:

$$-50A\sin(50t) + 50B\cos(50t) + 100(A\cos(50t) + B\sin(50t))$$
$$= 2\cos(50t)$$

$$(-50A + 100B)\sin 50t + (50B + 100A)\cos 50t = 2\cos 50t.$$

$-50A + 100B = 0,$ so $B = \frac{1}{2}A$

$50B + 100A = 2$ Now substitute for B.

$$50(\tfrac{1}{2}A) + 100A = 2$$

$$125A = 2$$

$$\Longrightarrow \quad A = \frac{2}{125} \quad \Longrightarrow \quad B = \frac{1}{125}.$$

$$\Longrightarrow \quad Q_p = \frac{2}{125}\cos(50t) + \frac{1}{125}\sin(50t).$$

$$Q(t) = Q_c(t) + Q_p(t) = c_1 e^{(-100t)} + \frac{1}{125}(2\cos(50t) + \sin(50t)).$$

$0 = Q(0) = c_1 + \frac{2}{125}$, so $c_1 = -\frac{2}{125}$.

$$Q(t) = \frac{1}{125}(-2e^{(-100t)} + 2\cos(50t) + \sin(50t)).$$

$$I(t) = Q'(t) = \frac{1}{125}(200e^{(-100t)} - 100\sin(50t) + 50\cos(50t))$$

$$I(t) = \frac{2}{5}\left(4e^{(-100t)} - 2\sin(50t) + \cos(50t)\right).$$

Ex. Now suppose $R = 10$ ohms, $L = 2$ H, $C = 0.01F$, $E(t) = 200\,V$, $I(0) = 0$, $Q(0) = 4$. Find $I(t)$.

$$LI'' + RI' + \frac{1}{C}I = E'(t) \qquad E(t) \text{ is a constant.}$$

$$2I'' + 10I' + \frac{1}{.01}I = 0$$

$$2I'' + 10I' + 100I = 0$$

$$I'' + 5I' + 50I = 0.$$

Characteristic equation:

$$r^2 + 5r + 50 = 0$$

$$r = \frac{-5 \pm \sqrt{25-200}}{2} = \frac{-5 \pm \sqrt{-175}}{2} = \frac{-5 \pm 5\sqrt{7}i}{2}$$

$$r = \frac{-5}{2} \pm \frac{5\sqrt{7}}{2}i.$$

$$\Longrightarrow \quad I(t) = e^{(-\frac{5}{2}t)}\left(c_1 \cos\frac{5\sqrt{7}}{2}t + c_2 \sin\frac{5\sqrt{7}}{2}t\right).$$

$$I'(t) = e^{(-\frac{5}{2}t)}\left(-\frac{5\sqrt{7}}{2}c_1 \sin\frac{5\sqrt{7}}{2}t + \frac{5\sqrt{7}}{2}c_2 \cos\frac{5\sqrt{7}}{2}t\right)$$
$$-\frac{5}{2}e^{(-\frac{5}{2}t)}\left(c_1 \cos\frac{5\sqrt{7}}{2}t + c_2 \sin\frac{5\sqrt{7}}{2}t\right).$$

$$0 = I(0) = 1(c_1 + 0), \quad \text{so } c_1 = 0.$$

We can get $I'(0)$ from:

$$LI'(0) + RI(0) + \frac{1}{C}Q(0) = E(0).$$

$$2(I'(0)) + 10(0) + \frac{1}{.01}(4) = 200$$

$$2I'(0) = -200$$

$$I'(0) = -100.$$

But $\quad I'(0) = \frac{5\sqrt{7}}{2}c_2 - \frac{5}{2}c_1 = \frac{5\sqrt{7}}{2}c_2,$

so $\quad -100 = \frac{5\sqrt{7}}{2}c_2$

$\Longrightarrow \quad c_2 = -\frac{40}{\sqrt{7}}$

Thus we have:

$$I(t) = e^{(-\frac{5}{2}t)}\left(-\frac{40}{\sqrt{7}}\sin\frac{5\sqrt{7}}{2}t\right).$$