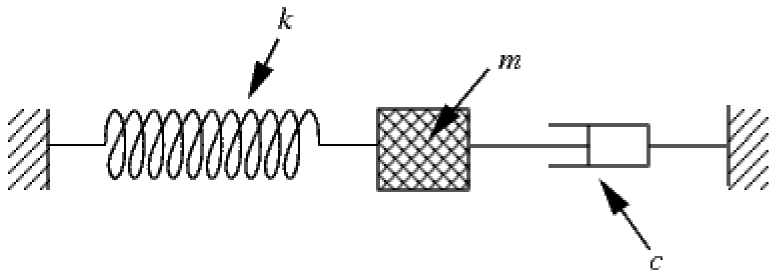
Forced Oscillations and Resonance

In the section on Vibrating Springs we considered a mass, m, attached to a spring on one end and a dashpot (like a shock absorber) on the other.



If $x(t)$ is the position of the mass at time t, we were led to the differential equation:

$$mx'' + cx' + kx = F(t),$$

where, k, is the spring constant, c, is the constant coming from the force of the dashpot, and $F(t)$ is an external force.

In the section Vibrating Springs we only considered the situation where $F(t) = 0$. In that case we say the motion is "free". If $F(t) \neq 0$ we say the motion is "forced".

In this section we will consider the situation where the motion is forced and the external force, $F(t)$, is a simple harmonic function given by $F(t) = F_0 \cos(wt)$ (we could also have used $F(t) = F_0 \sin(wt)$).

Undamped Forced Oscillations

Recall that in the absence of a dashpot (i.e., $c = 0$), we called the motion "undamped". In the section on Vibrating Springs this led us to solving the differential equation:

$$mx'' + kx = 0.$$

In this section we will consider a system with a spring, no dashpot (so the motion is "undamped") and an external force of the form $F(t) = F_0 \cos(wt)$. Thus we need to solve:

$$mx'' + kx = F_0 \cos(wt).$$

To solve this non-homogeneous differential equation, we need to first solve the homogenous equation:

$$mx'' + kx = 0.$$

Recall that when we solved this equation earlier, we let $w_0 = \sqrt{\frac{k}{m}}$, so our equation became:

$$x'' + w_0^2 x = 0$$

whose general solution is:

$$x(t) = Acos(w_0 t) + Bsin(w_0 t).$$

We then found that we could write the general solutions as:

$$x(t) = Ccos(w_0 t - \alpha), \quad \textit{where } C = \sqrt{A^2 + B^2},\ \tan(\alpha) = \frac{B}{A}.$$

So when solving:

$$mx'' + kx = F_0 \cos(wt)$$

We start with the general solution to the homogenous equation:

$$mx'' + kx = 0 \qquad \textit{or} \qquad x'' + w_0^2 x = 0,$$

which is $x_c(t) = Ccos(w_0 t - \alpha)$ *(Note: in general* $w \neq w_0$*).*

We now need to find a particular solution, x_p, and the general solution is $x(t) = x_c(t) + x_p(t)$.

Since $F(t) = F_0 \cos(wt)$, In general, we might expect the particular solution to have the form:

$$x_p = Ecos(wt) + Fsin(wt).$$

However, in the case where $c = 0$ (ie the undamped case), F will always turn out to be 0, so we can try:

$$x_p = Ecos(wt)$$

$$x_p' = -Ewsin(wt)$$

$$x_p'' = -Ew^2 \cos(wt).$$

Plugging into $mx'' + kx = F_0 \cos(wt)$, we get:

$$-mE^2 \mathrm{w}^2 \cos(wt) + kEcos(wt) = F_0 \cos(wt)$$

$$Ecos(wt)[-mw^2 + k] = F_0 \cos(wt)$$

$$E = \frac{F_0}{k - mw^2} = \frac{F_0}{m\left(\frac{k}{m} - w^2\right)}.$$

Since $w_0 = \sqrt{\frac{k}{m}}$, $w_0^2 = \frac{k}{m}$, *so the expression for E becomes:*

$$E = \frac{F_0}{m\left(w_0^2 - w^2\right)}.$$

Notice at $w = w_0$, E is undefined.

So the general solution becomes:

$$x(t) = Ccos(w_0 t - \alpha) + \frac{F_0}{m\left(w_0^2 - w^2\right)} \cos(wt).$$

Notice that as w goes to w_0 , the amplitude of the oscillations of an undamped system increases without bound. This is called resonance.

Ex. Given $m = \frac{1}{2}$, $c = 0$, $k = 4, w = 4$, $F_0 = 40$, $x(0) = 2$, *and* $x'(0) = 0$, *solve*

$$mx'' + cx' + kx = F_0 \cos(wt).$$

In this case the differential equation becomes:

$$\frac{1}{2}x'' + 4x = 40\cos(4t) \qquad or \qquad x'' + 8x = 80\cos(4t).$$

From an example we did in the section on Vibrating Springs, we know the general solutions to $x'' + 8x = 0$ *is:*

$$x(t) = ACos\left(2\sqrt{2}t\right) + Bsin\left(2\sqrt{2}t\right).$$

Thus for $x'' + 8x = 80\cos(4t)$ *we have:*

$$x_c(t) = ACos\left(2\sqrt{2}t\right) + Bsin\left(2\sqrt{2}t\right).$$

To find a particular solution, since $c = 0$ *we can try:*

$$x_p = E\cos(4t)$$

$$x_p' = -4E\sin(4t)$$

$$x_p'' = -16E\cos(4t).$$

Plugging into $x'' + 8x = 80\cos(4t)$ *we get:*

$$-16Ecos(4t) + 8Ecos(4t) = 80\cos(4t)$$

$$-8Ecos(4t) = 80\cos(4t) \qquad \Rightarrow \quad E = -10.$$

Thus, the general solution to $x'' + 8x = 80\cos(4t)$ *is:*

$$x(t) = ACos\left(2\sqrt{2}t\right) + Bsin\left(2\sqrt{2}t\right) - 10\cos(4t).$$

The initial conditions are: $x(0) = 2, \quad x'(0) = 0.$

$$2 = x(0) = A - 10 \quad \Rightarrow \quad A = 12.$$

$$x'(t) = -2\sqrt{2}\,\mathrm{Asin}\left(2\sqrt{2}t\right) + 2\sqrt{2}B\cos\left(2\sqrt{2}t\right) + 40\sin(4t)$$

$$0 = x'(0) = 2\sqrt{2}B \quad \Rightarrow \quad B = 0.$$

$$x(t) = 12\cos\left(2\sqrt{2}t\right) - 10\cos(4t).$$

Damped Forced Oscillations

In the case of damped forced oscillations we need to solve:

$$mx'' + cx' + kx = F_0\cos(wt).$$

As usual, we start by solving the homogenous equation:

$$mx'' + cx' + kx = 0.$$

As we saw in the section on Vibrating Springs there are 3 cases:

1. *Overdamped case:* $c^2 - 4km > 0$, *2 real roots* r_1, r_2, *both negative;*
$$x(t) = Ae^{r_1 t} + Be^{r_2 t}.$$

2. *Critically damped case:* $c^2 - 4km = 0$, r_1, r_2, *2 equal negative roots;*
$$x(t) = e^{rt}(A + Bt).$$

3. *Underdamped case:* $c^2 - 4km < 0$, $r = a \pm bi$, $a < 0$;
$$x(t) = e^{at}(c_1 \cos(bt) + c_2 \sin(bt)).$$

Notice in each case $\lim_{t\to\infty} x(t) = 0$ *so for the case when:*
$$mx'' + cx' + kx = F_0 \cos(wt)$$

and $x(t) = x_c(t) + x_p(t)$, *we have* $\lim_{t\to\infty} x_c(t) = 0$.
Thus $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x_p(t)$, *that is, the general solution approaches the particular solution.*

Ex. Suppose $m = \frac{1}{2}$, $c = 3$, $k = 4$, $w = 4$, $F_0 = 40$, $x(0) = 2$, *and* $x'(0) = 0$.
Solve: $\frac{1}{2}x'' + 3x' + 4x = 40\cos(4t)$ *or* $x'' + 6x' + 8x = 80\cos(4t)$.

From an example in "Vibrating Springs" we know the solutions to:
$$x'' + 6x' + 8x = 0$$

are given by: $$x(t) = Ae^{-2t} + Be^{-4t}.$$

Now let's find a particular solution.
Since $c \neq 0$, we need to try:

$$x_p = E\cos(4t) + Fsin(4t)$$
$$x_p' = -4Esin(4t) + 4Fcos(4t)$$
$$x_p'' = -16Ecos(4t) - 16Fsin(4t).$$

Substituting into $x'' + 6x' + 8x = 80\cos(4t)$, we get:

$$-16Ecos(4t) - 16Fsin(4t) - 24Esin(4t) + 24Fcos(4t)$$
$$+8Ecos(4t) + 8Fsin(4t) = 80\cos(4t).$$

$$(-16E + 24F + 8E)\cos(4t)$$
$$+(-16F - 24E + 8F)\sin(4t) = 80\cos(4t).$$

$$-8E + 24F = 80$$
$$-24E - 8F = 0 \quad \Rightarrow \quad F = -3E.$$

$$-8E - 72E = 80 \quad \Rightarrow \quad E = -1, \quad F = 3.$$

So $x_p = -\cos(4t) + 3\sin(4t)$.

Thus the general solution is:

$$x(t) = Ae^{-2t} + Be^{-4t} - \cos(4t) + 3\sin(4t).$$

To solve for A and B we use: $x(0) = 2$ *and* $x'(0) = 0$.

$x'(t) = -2Ae^{-2t} - 4Be^{-4t} + 4\sin(4t) + 12\cos(4t)$, *so we have:*

$$2 = x(0) = A + B - 1 \qquad \Rightarrow \qquad A + B = 3$$
$$0 = x'(0) = -2A - 4B + 12 \qquad \Rightarrow \qquad A + 2B = 6.$$

Solving these simultaneous equations we get: $B = 3, \;\; A = 0.$

Thus the solution is:

$$x(t) = 3e^{-4t} - \cos(4t) + 3\sin(4t).$$

Ex. Solve $mx'' + cx' + kx = F_0 \cos(wt)$ *where:*

a. $m = 1, \; c = 0, \; k = 125, \; w = 5, \; F_0 = 50, \; x(0) = 6,$ *and* $v(0) = 50.$
b. $m = 1, \; c = 10, \; k = 125, \; w = 5, \; F_0 = 50, \; x(0) = 6,$ *and* $v(0) = 50.$

a. *We need to solve:* $x'' + 125x = 50\cos(5t), \;\; x(0) = 6, \;\; x'(0) = 50.$

From an example in "Vibrating Springs" we know the solutions of:

$$x'' + 125x = 0$$

are given by: $x(t) = Acos\left(5\sqrt{5}t\right) + Bsin\left(5\sqrt{5}t\right).$

Since $c = 0$*, to find a particular solution we try:*

$$x_p = Ecos(5t)$$

$$x_p' = -5Esin(5t)$$

$$x_p'' = -25Ecos(5t).$$

Plugging into $x'' + 125x = 50\cos(5t)$ *we get:*

$$-25E cos(5t) + 125E cos(5t) = 50\cos(5t)$$

$$100\,\mathrm{E}\cos(5t) = 50\cos(5t) \qquad \Rightarrow E = \frac{1}{2}.$$

Thus $x_p = \frac{1}{2} cos(5t).$

So the general solution to $x'' + 125x = 50\cos(5t)$ *is:*

$$x(t) = A cos\left(5\sqrt{5}t\right) + B sin\left(5\sqrt{5}t\right) + \frac{1}{2}\cos(5t).$$

To solve for A *and* B*, we use:* $x(0) = 6,\ \ x'(0) = 50.$

$$x'(t) = -5\sqrt{5}\,\mathrm{A}\sin\left(5\sqrt{5}t\right) + 5\sqrt{5}B cos\left(5\sqrt{5}t\right) - \frac{5}{2}\sin(5t)$$

So we have:

$$6 = x(0) = A + \frac{1}{2} \qquad \Rightarrow \qquad A = \frac{11}{2}$$

$$50 = x'(0) = 5\sqrt{5}B \qquad \Rightarrow \qquad B = \frac{10}{\sqrt{5}}.$$

Thus the solution to $x'' + 125x = 50\cos(5t)$, $x(0) = 6$, $x'(0) = 50$ *is:*

$$x(t) = \frac{11}{2} cos\left(5\sqrt{5}t\right) + \frac{10}{\sqrt{5}} sin\left(5\sqrt{5}t\right) + \frac{1}{2}\cos(5t).$$

b. *We need to solve:*

$$x'' + 10x' + 125x = 50\cos(5t)\,,\ \ x(0) = 6,\ \ x'(0) = 50.$$

From an earlier example in Vibrating Springs we know the general solution to $x'' + 10x' + 125x = 0$ *is given by:*

$$x(t) = e^{-5t}\big(A cos(10t) + B\ sin(10t)\big).$$

Since $c \neq 0$*, the particular solution to* $x'' + 10x' + 125x = 50\cos(5t)$ *is of the form:*

$$x_p = E cos(5t) + F sin(5t)$$
$$x_p' = -5E sin(5t) + 5F cos(5t)$$
$$x_p'' = -25E cos(5t) - 25F sin(5t).$$

Plugging into $x'' + 10x' + 125x = 50\cos(5t)$ *we get:*

$$-25E cos(5t) - 25F sin(5t) + 10(-5E sin(5t) + 5F cos(5t)) \\ +125\big(E cos(5t) + F sin(5t)\big) = 50\cos(5t).$$

$$(-25E + 50F + 125E) cos(5t) \\ + (-25F - 50E + 125F)\sin(5t) = 50\cos(5t).$$

$$100E + 50F = 50$$
$$-50E + 100F = 0 \qquad \Rightarrow \qquad E = 2F.$$

$$200F + 50F = 50 \qquad \Rightarrow \qquad F = \tfrac{1}{5}\,,\quad E = \tfrac{2}{5}.$$

So $x_p = \frac{2}{5}\cos(5t) + \frac{1}{5}\sin\,(5t).$

So the general solution to $x'' + 10x' + 125x = 50\cos(5t)$ *is:*

$$x(t) = e^{-5t}\big(A cos(10t) + B\ sin(10t)\big) + \frac{2}{5}\cos(5t) + \frac{1}{5}\sin(5t).$$

To find A *and* B *we use:* $x(0) = 6,\quad x'(0) = 50.$

$$x'(t) =$$

$$e^{-5t}\big(-10A sin(10t) + 10B cos(10t)\big)$$
$$- 5e^{-5t}\big(A cos(10t) + B\ sin(10t)\big) - 2\sin(5t) + \cos(5t).$$

$$6 = x(0) = A + \frac{2}{5} \qquad \Rightarrow \qquad A = \frac{28}{5}.$$

$$50 = x'(0) = 10B - 5A + 1$$

$$50 = 10B - 5\left(\frac{28}{5}\right) + 1$$

$$50 = 10B - 27$$

$$77 = 10B \qquad \Rightarrow \qquad B = \frac{77}{10}.$$

Thus the solutions is:

$$x(t) = e^{-5t}\left(\frac{28}{5} cos(10t) + \frac{77}{10}\ sin(10t)\right) + \frac{2}{5}\cos(5t) + \frac{1}{5}\sin(5t).$$