Integrating Differential Equations

A **differential equation** is an equation relating a function and one or more of its derivatives. The solutions to a differential equation are all functions that satisfy the equation.

Differential Equation	General Solution
$\frac{dy}{dx} = 2x$	$y = x^2 + c$
$y'' + 9y = 0$	$y = A\cos 3x + B\sin 3x$

In the second case notice that:

$$y = A\cos 3x + B\sin 3x$$

$$y' = -3A\sin 3x + 3B\cos 3x$$

$$y'' = -9A\cos 3x - 9B\sin 3x$$

Thus we have:

$$y'' + 9y = (-9A\cos 3x - 9B\sin 3x) + 9(A\cos 3x + B\sin 3x) = 0.$$

Hence $y = A\cos 3x + B\sin 3x$ is a solution to the differential equation

$$y'' + 9y = 0.$$

$y'' + 9y = 0$ is called a 2nd order differential equation because the second deriative of y appears in the equation.

Initial Value Problem: The solution to a differential equation given an initial condition or conditions.

Ex. Solve $\frac{dy}{dx} = 4x^3$; with the initial condition that $y(1) = 3$.

General Solution: $y = x^4 + c$

Initial Condition: $y(1) = 1^4 + c = 3$

$\Rightarrow c = 2$

$y = x^4 + 2$ is called the **Particular Solution**.

$y = x^4 + c$ is called the **General Solution**.

If we have $\frac{dy}{dx} = g(x)$ or $\frac{d^2y}{dx^2} = g(x)$ we can try integrating both sides.

Ex. Solve $\frac{dy}{dx} = x^2\sqrt{x^3 + 1}$; $y(2) = 3$.

General Solution:

$$y = \int x^2 (x^3 + 1)^{\frac{1}{2}} dx$$

Let $u = x^3 + 1$

$$du = 3x^2 dx$$

$$\frac{1}{3} du = x^2 dx$$

$$\int x^2 (x^3 + 1)^{\frac{1}{2}} dx = \frac{1}{3} \int u^{\frac{1}{2}} du = \frac{1}{3}\left(\frac{2}{3} u^{\frac{3}{2}}\right) + c$$

$$\boxed{y = \frac{2}{9}(x^3 + 1)^{\frac{3}{2}} + c}$$

Particular Solution:

$$y(2) = \frac{2}{9}(2^3 + 1)^{\frac{3}{2}} + c = 3$$

$$\frac{2}{9}(9)^{\frac{3}{2}} + c = 3$$

$$\frac{2}{9}(27) + c = 3$$

$$6 + c = 3$$

$$c = -3$$

$$\boxed{y = \frac{2}{9}(x^3 + 1)^{\frac{3}{2}} - 3}$$

Ex. Find the General Solution of $\frac{d^2y}{dx^2} = 6x + \sin x$.

$$\frac{dy}{dx} = \int 6x + \sin x \, dx = 3x^2 - \cos x + c_1$$

$$y = \int (3x^2 - \cos x + c_1)\, dx = x^3 - \sin x + c_1 x + c_2.$$

Velocity and Acceleration

When a particle moves in a line and the position is given by $x(t)$ then,

$$v(t) = \frac{dx}{dt} = \text{velocity}$$

$$a(t) = \frac{d^2x}{dt^2} = \text{acceleration} = \frac{dv}{dt}.$$

Ex. A ball is dropped from 100 feet above the ground. If acceleration due to gravity is $-32ft/sec^2$ find the position after 2 seconds.

So we must solve the differential equation:

$\frac{d^2x}{dt^2} = -32$; subject to the initial conditions: $x(0) = 100,\ v(0) = 0.$

$$a = \frac{d^2x}{dt^2} = -32 = \frac{dv}{dt}$$

$$v(t) = \int -32dt = -32t + c$$

$$0 = v(0) = -32(0) + c \quad \Rightarrow c = 0.$$

$$v(t) = -32t.$$

$$v(t) = \frac{dx}{dt} = -32t$$

$$x(t) = \int -32t\ dt = -16t^2 + c$$

$$100 = x(0) = -16(0)^2 + c \quad \Rightarrow \quad c = 100$$

$$x(t) = -16t^2 + 100.$$

$$x(2) = -16(2)^2 + 100 = \boxed{36 \text{ feet above the ground.}}$$

Ex. A ball is launched straight up with an initial velocity of $96ft/sec$ from a point 256 feet above the water. If acceleration due to gravity is $-32ft/sec^2$, find the time t when the ball hits the water.

Differential equation: $a = \frac{d^2x}{dt^2} = -32 = \frac{dv}{dt}$

Initial Conditions: $x(0) = 256, \quad v(0) = 96.$

$a(t) = \frac{dv}{dt} = -32$

$v(t) = \int -32\, dt = -32t + c_1$

$96 = v(0) = -32(0) + c_1 \quad \Rightarrow c_1 = 96$ so

$v(t) = -32t + 96.$

$x(t) = \int v(t)\, dt = \int (-32t + 96) dt = -16t^2 + 96t + c_2$

$256 = x(0) = -16(0)^2 + 96(0) + c_2 \quad \Rightarrow c_2 = 256$ so

$x(t) = -16t^2 + 96t + 256.$

The ball hits the water when $x(t) = 0.$

$0 = x(t) = -16(t^2 - 6t - 16) = -16(t-8)(t+2) = 0$

$t = 8, -2$

Since $t \geq 0$, the ball hits the water after 8 seconds.