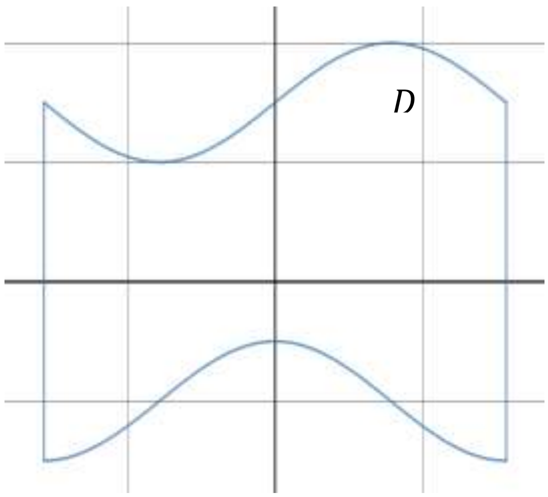
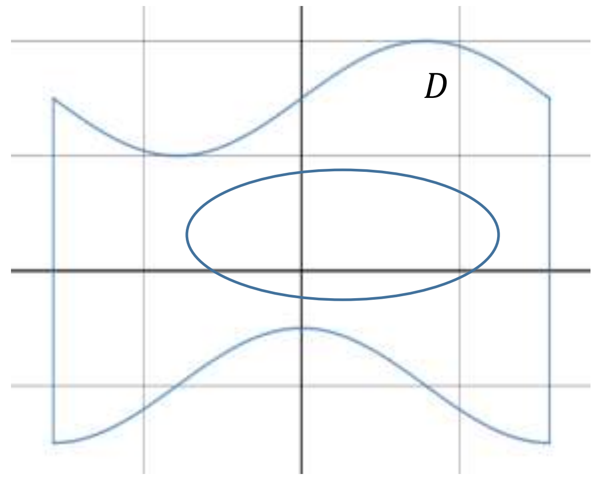
Cauchy's Theorem

Def. A **simply connected domain** D is one in which every simple closed contour within D encloses only points of D.



Simply Connected



Not Simply Connected

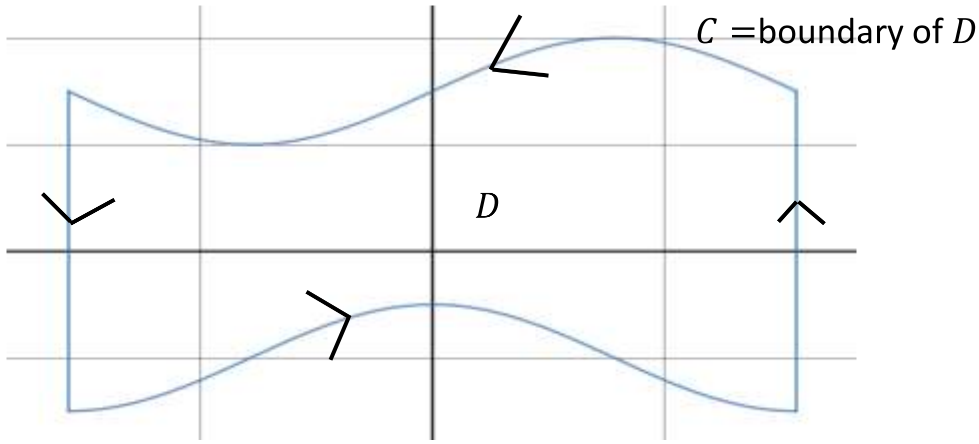
Cauchy's Theorem: If f is analytic and $f'(z)$ is continuous in a simply connected domain D, then along a simple closed curve C in D:

$$\oint_C f(z)dz = 0.$$

To prove this theorem we are going to use Green's Theorem.

Green's Theorem: Let the real valued functions $u(x, y)$, $v(x, y)$ along with their partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$, be continuous throughout a simply connected region D consisting of points interior to and on a simple closed contour C in the xy-plane. Let C be oriented in the positive direction, then:

$$\oint_C u(x, y)dx + v(x, y)dy = \iint_D \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy.$$



Proof of Cauchy's Theorem:

Let $f(z) = u(x,y) + iv(x,y)$ and $dz = dx + idy$ then:

$$\oint_C f(z)dz = \oint_C (u(x,y) + iv(x,y))(dx + idy)$$

$$= \oint_C udx - vdy + i\oint_C udy + vdx.$$

Since $f'(z)$ is continuous, so are the partial derivatives of u and v. Thus u, v satisfy Green's theorem.

$$\oint_C u(x,y)dx - v(x,y)dy = \iint_D (-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})dxdy$$

$$\oint_C u(x,y)dy + v(x,y)dx = \iint_D (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})dxdy.$$

So we have:

$$\oint_C f(z)dz = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_D (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})dxdy.$$

But since $f(z)$ is analytic in D, the Cauchy-Riemann equations hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

So both integrals on the RHS are 0 and

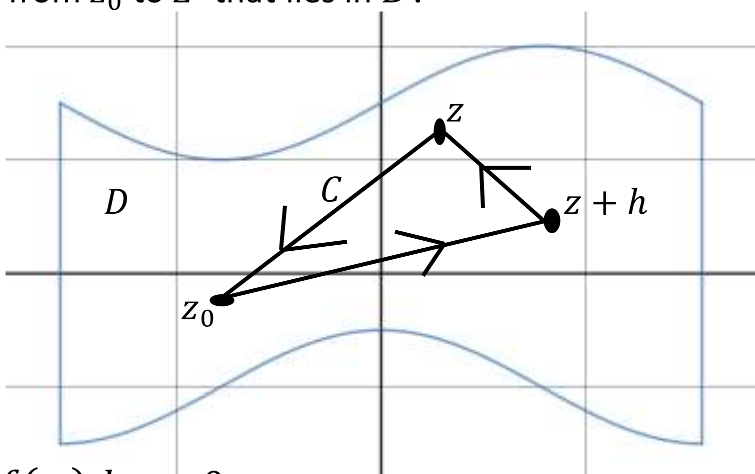
$$\oint_C f(z)dz = 0.$$

Theorem: If $f(z)$is continuous in a simply connected domain D and if $\oint_C f(z)dz = 0$ for every closed contour C lying in D, then there exists a function $F(z)$, analytic in D, such that $F'(z) = f(z)$.

Proof: For points inside of D define $F(z)$ as:

$$F(z) = \int_{z_0}^{z} f(w)dw.$$

where the integral is along a contour from z_0 to z that lies in D.



Since $\oint_C f(z)dz = 0$ for every closed contour C lying in D we have:

$$\int_{z_0}^{z+h} f(w)dw + \int_{z+h}^{z} f(w)dw + \int_{z}^{z_0} f(w)dw = 0.$$

Since $F(z) = \int_{z_0}^{z} f(w)dw \implies -F(z) = \int_{z}^{z_0} f(w)dw$. So:

$$F(z+h) + \int_{z+h}^{z} f(w)dw - F(z) = 0.$$

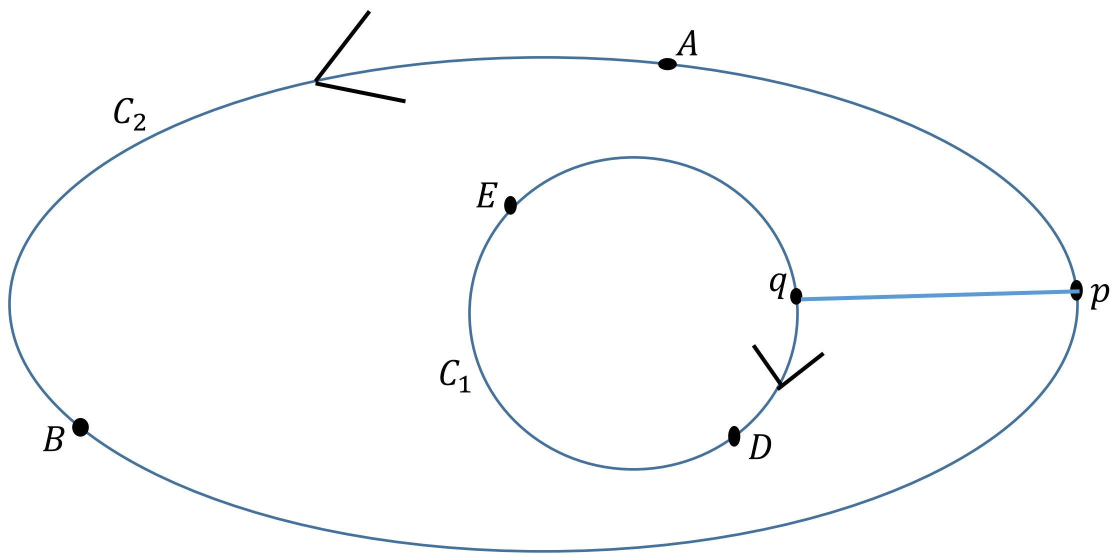
$$\implies \qquad F(z+h) - F(z) = \int_{z}^{z+h} f(w)dw.$$

Now divide both sides by h and take the limit as h goes to 0:

$$F'(z) = \lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = \lim_{h\to 0} \frac{1}{h} \int_{z}^{z+h} f(w)dw = f(z).$$

Notice the RHS equals $f(z)$ by the fundamental theorem of calculus.

Cauchy's theorem can be extended to multiply connected domains by using crosscuts. Suppose we have a multiply connected region which is bounded by two simple curves C_1 and C_2 (where C_1 is inside of C_2). Choose a point $p \in C_2$ and $q \in C_1$ and draw the line segment as shown below.



The region bounded by $ABpqDEqpA$ is simply connected so Cauchy's theorem applies. So if f is analytic in the region bounded by $ABpqDEqpA$ then:

$$\int_{pABpqDEqp} f(z)dz = 0.$$

So we have:

$$\int_{C_2} f(z)dz + \int_p^q f(z)dz + \int_{-C_1} f(z)dz + \int_q^p f(z)dz = 0.$$

But since: $\int_p^q f(z)dz = -\int_q^p f(z)dz,$ we have:

$$\int_{C_2} f(z)dz + \int_{-C_1} f(z)dz = 0 \implies \int_C f(z)dz = 0$$

where $C = C_2 - C_1$.

Also notice that since $\int_{-C_1} f(z)dz = -\int_{C_1} f(z)dz$ we get:

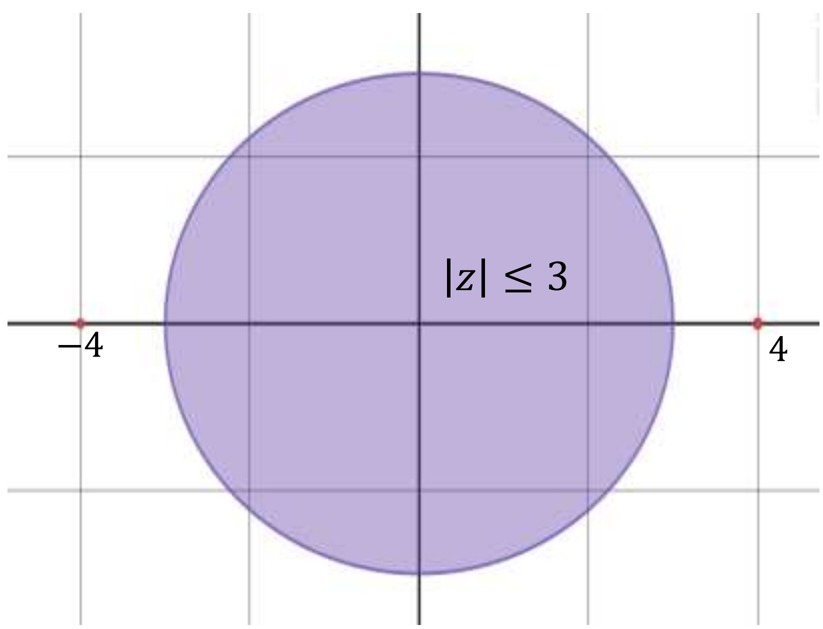
$$\int_{C_2} f(z)dz - \int_{C_1} f(z)dz = 0$$

$$\int_{C_2} f(z)dz = \int_{C_1} f(z)dz.$$

This last relationship will turn out to be very useful.

Ex. Evaluate $\oint_C \frac{e^z}{z^2-25} dz$; where C is the circle $|z| = 3$.

The only non-analytic points of $f(z) = \frac{e^z}{z^2-25}$ are at $z = \pm 5$, which are not inside the simply connected region $|z| \leq 3$ (disk of radius 3). Thus $f(z)$ is analytic in this disk and $\oint_C \frac{e^z}{z^2-25} dz = 0$ by Cauchy's theorem.



Ex. Evaluate $\oint_C \frac{e^z}{(z^2-25)z}dz$; where C is the boundary of the annulus between the circles $|z|=1$ and $|z|=3$.

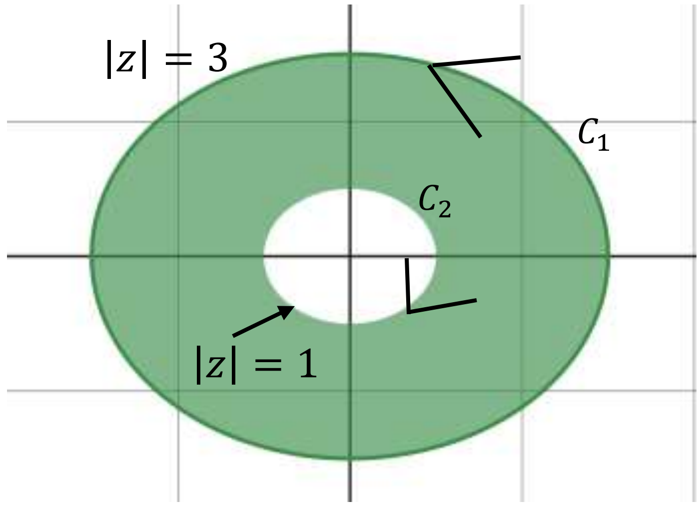
$f(z)=\frac{e^z}{(z^2-25)z}$ is analytic

in the annulus $1 \le |z| \le 3$ so

$\oint_C \frac{e^z}{(z^2-25)z}dz =$

$$\oint_{C_1-C_2} \frac{e^z}{(z^2-25)z}dz = 0.$$

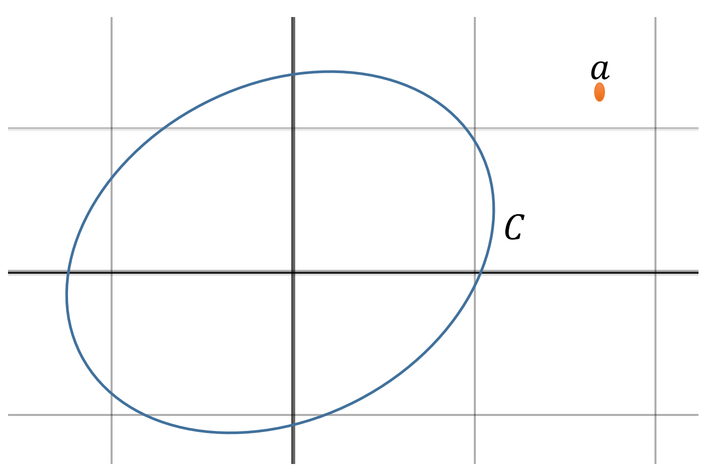
by Cauchy's theorem.



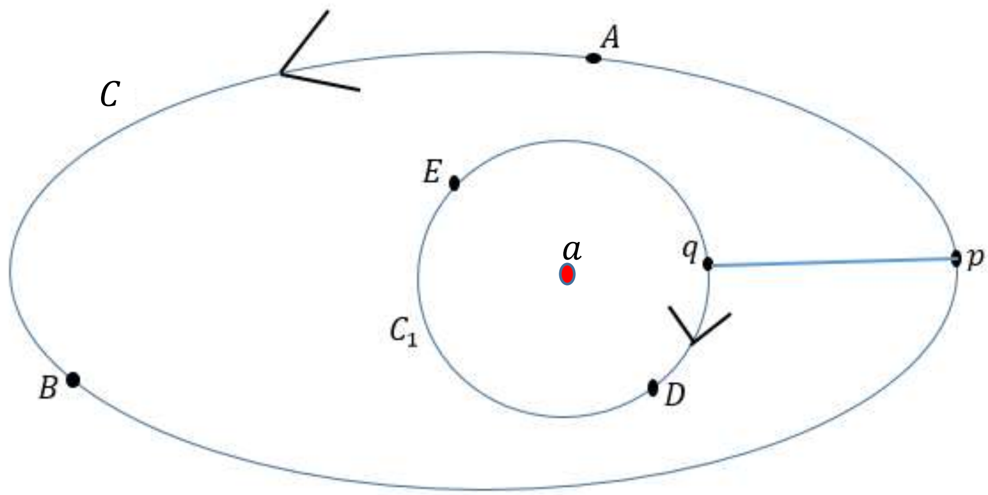
Ex. Evaluate $\frac{1}{2\pi i}\oint_C \frac{dz}{(z-a)^m}$, $m=1,2,3,\ldots$, and C is a simple closed contour with $a \notin C$.

There are 2 cases: 1. where $z=a$ is outside of C; 2. where $z=a$ is inside C.

Case 1: If $z=a$ is outside of C then $f(z)=\frac{1}{(z-a)^m}$; $m=1,2,3,\ldots$, is analytic inside the region bounded by C and so $\frac{1}{2\pi i}\oint_C \frac{dz}{(z-a)^m}=0$, by Cauchy's theorem.



Case 2: If $z = a$ is inside of C then we can create a crosscut to a circle, C_1, whose center is $z = a$ and lies inside of C.



Since $f(z) = \frac{1}{(z-a)^m}$; $m = 1,2,3,\ldots$, is analytic in the region bounded by C and C_1 we have:

$\oint_{pABpqDEp} \frac{dz}{(z-a)^m} = 0$ by Cauchy's theorem. But

$$\oint_{pABpqDEp} \frac{dz}{(z-a)^m} = \oint_C \frac{dz}{(z-a)^m} + \int_p^q \frac{dz}{(z-a)^m} + \oint_{-C_1} \frac{dz}{(z-a)^m} + \int_q^p \frac{dz}{(z-a)^m}$$
$$= 0.$$

However, $\int_p^q \frac{dz}{(z-a)^m} + \int_q^p \frac{dz}{(z-a)^m} = 0$ so

$$\oint_C \frac{dz}{(z-a)^m} + \oint_{-C_1} \frac{dz}{(z-a)^m} = 0 \Rightarrow \quad \oint_C \frac{dz}{(z-a)^m} = \oint_{C_1} \frac{dz}{(z-a)^m}$$

So $\frac{1}{2\pi i}\oint_C \frac{dz}{(z-a)^m} = \frac{1}{2\pi i}\oint_{C_1} \frac{dz}{(z-a)^m} = 0 \quad if\ m \neq 1$ (from earlier example)

$$= 1 \quad if\ m = 1.$$

Notice by using a crosscut we were able to turn a contour integral over a general simple closed contour into one over a circle (which is easier to calculate).

Ex. Evaluate $\frac{1}{2\pi i}\oint_C \frac{P'(z)}{P(z)}dz$ if $P(z)$ is a polynomial of degree n, with n simple (i.e. distinct) roots, none of which lie on a simple closed contour C.

Since $P(z)$ has n distinct roots we can factor it as:

$$P(z) = M(z-a_1)(z-a_2)(z-a_3)\dots(z-a_n)$$

where M is a constant and $a_1, a_2, a_3, \dots, a_n$ are the roots of $P(z)$.

Notice that

$$\frac{P'(z)}{P(z)} = \frac{d}{dz}Log(\,P(z))$$

$$= \frac{d}{dz}(Log(M(z-a_1)(z-a_2)(z-a_3)\dots(z-a_n))$$

$$= \frac{d}{dz}\big(Log(M) + Log(z-a_1) + Log(z-a_2) + \cdots Log(z-a_n)\big)$$

$$= \frac{1}{z-a_1} + \frac{1}{z-a_2} + \frac{1}{z-a_3} + \cdots + \frac{1}{z-a_n}$$

In the previous example we saw:

$$\frac{1}{2\pi i}\oint_C \frac{1}{z-z_0}dz = 0 \quad \text{if } z_0 \text{ is outside of } C$$

$$= 1 \quad \text{if } z_0 \text{ is inside of } C$$

Thus, $\frac{1}{2\pi i}\oint_C \frac{P'(z)}{P(z)}dz = \frac{1}{2\pi i}\oint_C \left(\frac{1}{z-a_1} + \frac{1}{z-a_2} + \frac{1}{z-a_3} + \cdots + \frac{1}{z-a_n}\right)dz$

$=$number of roots inside C.

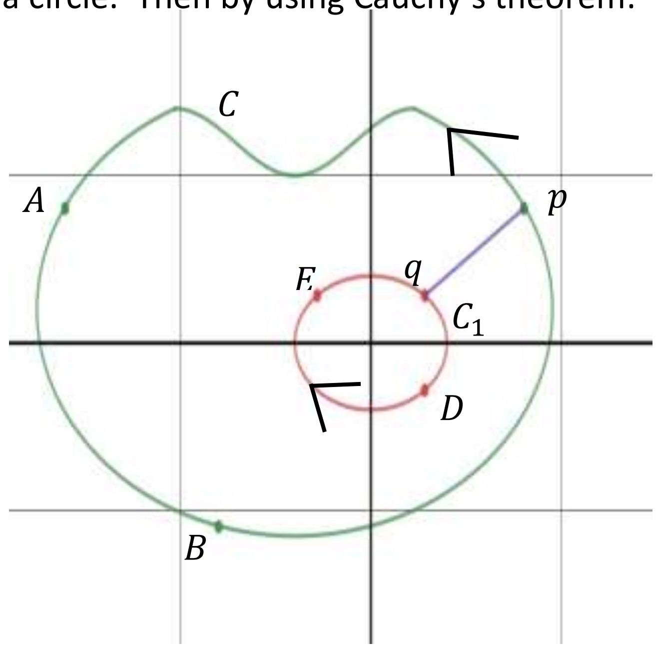
Ex. Evaluate $\oint_C f(z)dz$ where C is a simple closed contour where $z = 0$ is inside C and:

a. $f(z) = \dfrac{e^{(z^3)}}{z}$

b. $f(z) = \dfrac{e^{(z^3)}}{z^3}$

c. $f(z) = \dfrac{e^{(z^3)}}{z^4}$

a. First use a crosscut to turn the integral around a general simple closed contour into one that's a circle. Then by using Cauchy's theorem:



$$0 = \oint_{pABpqDEqp} \frac{e^{(z^3)}}{z} dz$$

$$= \oint_C \frac{e^{(z^3)}}{z} dz + \int_p^q \frac{e^{(z^3)}}{z} dz + \oint_{-C_1} \frac{e^{(z^3)}}{z} dz + \int_q^p \frac{e^{(z^3)}}{z} dz.$$

Notice that $\int_p^q \frac{e^{(z^3)}}{z}dz + \int_q^p \frac{e^{(z^3)}}{z}dz = 0$; so

$$0 = \oint_C \frac{e^{(z^3)}}{z}dz + \oint_{-C_1} \frac{e^{(z^3)}}{z}dz = \oint_C \frac{e^{(z^3)}}{z}dz - \oint_{C_1} \frac{e^{(z^3)}}{z}dz$$

So we have: $\oint_C \frac{e^{(z^3)}}{z}dz = \oint_{C_1} \frac{e^{(z^3)}}{z}dz.$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots \quad \text{so}$$

$$e^{(z^3)} = 1 + z^3 + \frac{z^6}{2!} + \frac{z^9}{3!} + \cdots + \frac{z^{3n}}{n!} + \cdots$$

$$\oint_{C_1} \frac{e^{(z^3)}}{z}dz = \oint_{C_1} \frac{1+z^3+\frac{z^6}{2!}+\frac{z^9}{3!}+\cdots+\frac{z^{3n}}{n!}+\cdots}{z}dz$$

$$= \oint_{C_1} \frac{1}{z}dz + \oint_{C_1} z^2 dz + \oint_{C_1} \frac{z^5}{2!}dz + \cdots \oint_{C_1} \frac{z^{3n-1}}{n!}dz + \cdots$$

All of the integrands except the first one are analytic inside the circle C_1 and therefore their integrals around C_1 are 0 by Cauchy's theorem.

The first integral we know is: $\oint_{C_1} \frac{1}{z}dz = 2\pi i$. Thus we have:

$$\oint_C \frac{e^{(z^3)}}{z}dz = \oint_{C_1} \frac{e^{(z^3)}}{z}dz = 2\pi i\ .$$

b. By a similar argument to part "a" (which you should make!!)

$$\oint_C \frac{e^{(z^3)}}{z^3}\,dz = \oint_{C_1} \frac{e^{(z^3)}}{z^3}\,dz.$$

Again using the power series for $e^{(z^3)}$ we get:

$$\oint_{C_1} \frac{e^{(z^3)}}{z^3}\,dz = \oint_{C_1} \frac{1+z^3+\frac{z^6}{2!}+\frac{z^9}{3!}+\cdots+\frac{z^{3n}}{n!}+\cdots}{z^3}\,dz$$

$$= \oint_{C_1} \frac{1}{z^3}\,dz + \oint_{C_1} 1\,dz + \oint_{C_1} \frac{z^3}{2!}\,dz + \cdots \oint_{C_1} \frac{z^{3n-3}}{n!}\,dz + \cdots$$

All of these integrals are 0 since $\oint_{C_1} z^n dz = 0,\ \ n \neq -1$, Thus we have:

$$\oint_C \frac{e^{(z^3)}}{z^3}\,dz = \oint_{C_1} \frac{e^{(z^3)}}{z^3}\,dz = 0.$$

c. Similarly:

$$\oint_C \frac{e^{(z^3)}}{z^4}\,dz = \oint_{C_1} \frac{e^{(z^3)}}{z^4}\,dz\ .$$

$$\oint_{C_1} \frac{e^{(z^3)}}{z^4}\,dz = \oint_{C_1} \frac{1+z^3+\frac{z^6}{2!}+\frac{z^9}{3!}+\cdots+\frac{z^{3n}}{n!}+\cdots}{z^4}\,dz$$

$$= \oint_{C_1} \frac{1}{z^4}\,dz + \oint_{C_1} \frac{1}{z}\,dz + \oint_{C_1} \frac{z^2}{2!}\,dz + \cdots \oint_{C_1} \frac{z^{3n-4}}{n!}\,dz + \cdots$$

Since $\oint_{C_1} z^m dz = 0 \qquad if\ m \neq -1$

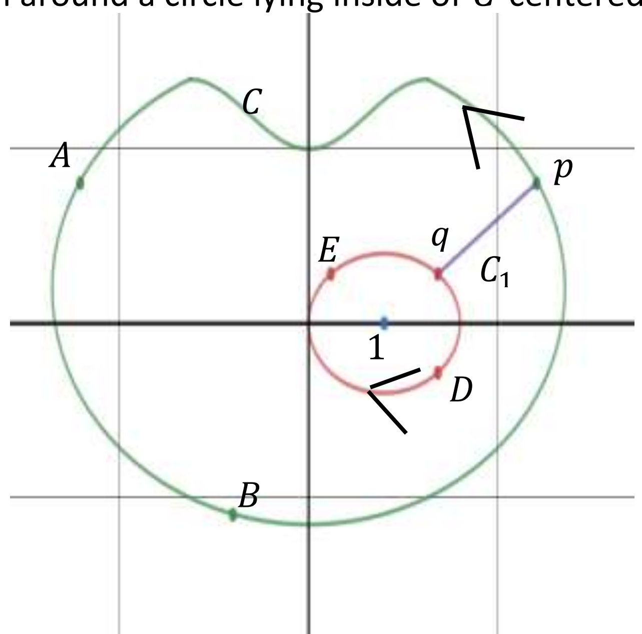
$$= 2\pi i \quad if\ m = -1$$

$\oint_{C_1} \frac{e^{(z^3)}}{z^4} dz = 2\pi i$. Thus

$$\oint_C \frac{e^{(z^3)}}{z^4} dz = \oint_{C_1} \frac{e^{(z^3)}}{z^4} dz = 2\pi i.$$

Ex. Evaluate $\oint_C \frac{e^{(z^3)}}{z-1} dz,$ where C is a simple closed contour and $z = 1$ is inside of C.

This problem looks a lot like part "a" of the previous example. Start by making a crosscut to turn this integral into an integral around a circle lying inside of C centered at $z = 1$.



$$0 = \oint_{pABpqDEqp} \frac{e^{(z^3)}}{z-1} dz$$

$$= \oint_C \frac{e^{(z^3)}}{z-1} dz + \int_p^q \frac{e^{(z^3)}}{z-1} dz + \oint_{-C_1} \frac{e^{(z^3)}}{z-1} dz + \int_q^p \frac{e^{(z^3)}}{z-1} dz.$$

Notice that $\int_p^q \frac{e^{(z^3)}}{z-1}dz + \int_q^p \frac{e^{(z^3)}}{z-1}dz = 0$; so

$$0 = \oint_C \frac{e^{(z^3)}}{z-1}dz + \oint_{-C_1} \frac{e^{(z^3)}}{z-1}dz = \oint_C \frac{e^{(z^3)}}{z-1}dz - \oint_{C_1} \frac{e^{(z^3)}}{z-1}dz.$$

So we have: $\oint_C \frac{e^{(z^3)}}{z-1}dz = \oint_{C_1} \frac{e^{(z^3)}}{z-1}dz.$

Now let $w = z - 1$, thus $w + 1 = z$.

$$\oint_{C_1} \frac{e^{(z^3)}}{z-1}dz = \oint_{C_1} \frac{e^{((w+1)^3)}}{w}dw.$$

$$= \oint_{C_1} \frac{1+(w+1)^3+\frac{(w+1)^6}{2!}+\frac{(w+1)^9}{3!}+\cdots+\frac{(w+1)^{3n}}{n!}+\cdots}{w}dw$$

$$= \oint_{C_1} \frac{1}{w}dw + \oint_{C_1} \frac{(w+1)^3}{w}dw + \oint_{C_1} \frac{(w+1)^6}{2!(w)}dw$$

$$+ \cdots + \oint_{C_1} \frac{(w+1)^{3n}}{n!(w)}dw + \cdots$$

Notice that $\frac{(w+1)^{3n}}{(w)} = \frac{1}{w} + \sum_{k=0}^{3n-1} d_k w^k$; so

$$\oint_{C_1} \frac{(w+1)^{3n}}{n!(w)}dw = \left(\frac{1}{n!}\right)2\pi i.$$

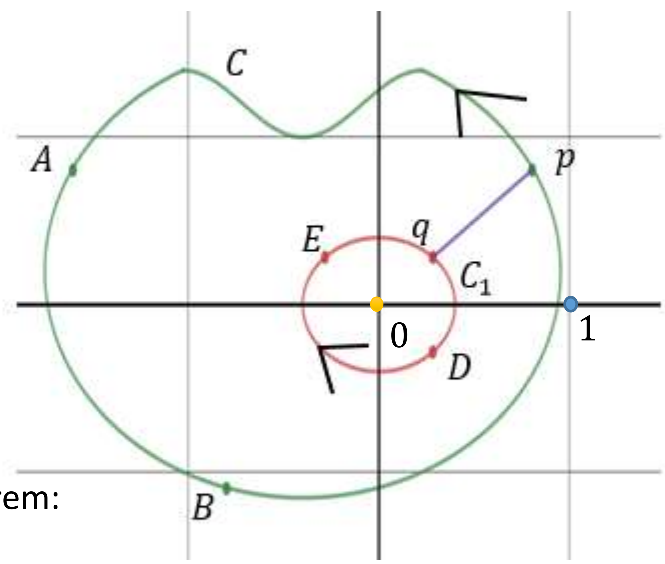
Thus we have:

$$\oint_{C_1} \frac{e^{(z^3)}}{z-1}\,dz = 2\pi i\left(\sum_{n=0}^{\infty}\frac{1}{n!}\right) = 2\pi e i \quad \text{and}$$

$$\oint_{C} \frac{e^{(z^3)}}{z-1}\,dz = \oint_{C_1} \frac{e^{(z^3)}}{z-1}\,dz = 2\pi e i.$$

Ex. Evaluate $\oint_C \frac{e^{(z^3)}}{z(z-1)}\,dz$ where C is a simple closed contour where $z = 0$ is inside of C and $z = 1$ is outside of C.

First use a crosscut to turn the integral around a general simple closed contour into one that's a circle.



Then by using Cauchy's theorem:

$$0 = \oint_{pABpqDEqp} \frac{e^{(z^3)}}{z(z-1)}\,dz$$

$$= \oint_{C} \frac{e^{(z^3)}}{z(z-1)}\,dz + \int_p^q \frac{e^{(z^3)}}{z(z-1)}\,dz + \oint_{-C_1} \frac{e^{(z^3)}}{z(z-1)}\,dz + \int_q^p \frac{e^{(z^3)}}{z(z-1)}\,dz.$$

Notice that $\int_p^q \frac{e^{\left(z^3\right)}}{z(z-1)}dz + \int_q^p \frac{e^{\left(z^3\right)}}{z(z-1)}dz = 0;$ so

$$0 = \oint_C \frac{e^{\left(z^3\right)}}{z(z-1)}dz + \oint_{-C_1} \frac{e^{\left(z^3\right)}}{z(z-1)}dz = \oint_C \frac{e^{\left(z^3\right)}}{z(z-1)}dz - \oint_{C_1} \frac{e^{\left(z^3\right)}}{z(z-1)}dz.$$

So we have: $$\oint_C \frac{e^{\left(z^3\right)}}{z(z-1)}dz = \oint_{C_1} \frac{e^{\left(z^3\right)}}{z(z-1)}dz.$$

C_1 is a circle of radius $R < 1$ and C_1 lies inside of C.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots$$ so

$$e^{\left(z^3\right)} = 1 + z^3 + \frac{z^6}{2!} + \frac{z^9}{3!} + \cdots + \frac{z^{3n}}{n!} + \cdots$$

$$\oint_{C_1} \frac{e^{\left(z^3\right)}}{z(z-1)}dz = \oint_{C_1} \frac{1+z^3+\frac{z^6}{2!}+\frac{z^9}{3!}+\cdots+\frac{z^{3n}}{n!}+\cdots}{z(z-1)}dz$$

$$= \oint_{C_1} \frac{1}{z(z-1)}dz + \oint_{C_1} \frac{z^2}{(z-1)}dz + \oint_{C_1} \frac{z^5}{(z-1)2!}dz + \cdots \oint_{C_1} \frac{z^{3n-1}}{(z-1)n!}dz + \cdots$$

All of the integrands except the first one are analytic inside the circle C_1 and therefore their integrals around C_1 are 0.

To evaluate $\oint_{C_1} \frac{1}{z(z-1)} dz$ we use partial fractions.

$\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$

$\frac{1}{z(z-1)} = \frac{A(z-1)+B(z)}{z(z-1)}$; Thus $1 = A(z-1) + B(z)$.

At $z = 1$ this becomes $1 = B$ and at

$z = 0$ this becomes $1 = -A$ or $A = -1$;

Thus $A = -1$, $B = 1$.

so $\frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$.

$$\oint_{C_1} \frac{1}{z(z-1)} dz = \oint_{C_1} \left(-\frac{1}{z} + \frac{1}{z-1}\right) dz = -\oint_{C_1} \frac{1}{z} dz + \oint_{C_1} \frac{1}{z-1} dz$$

The far right integral is 0 by Cauchy's theorem since $\frac{1}{z-1}$ is analytic inside C_1, a circle of radius $R < 1$.

We've already seen that $\oint_{C_1} \frac{1}{z} dz = 2\pi i$.

Thus: $\oint_{C} \frac{e^{(z^3)}}{z(z-1)} dz = \oint_{C_1} \frac{e^{(z^3)}}{z(z-1)} dz = -2\pi i$.